

LA-5061-MS

INFORMAL REPORT

C.3

CIC-14 REPORT COLLECTION
**REPRODUCTION
COPY**

A Monte Carlo Sampler

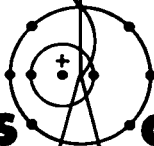
L

LOS ALAMOS NATIONAL LABORATORY



3 9338 00397 4259

J



los alamos
scientific laboratory

of the University of California

LOS ALAMOS, NEW MEXICO 87544



This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

In the interest of prompt distribution, this LAMS report was not edited by the Technical Information staff.

Printed in the United States of America. Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.95

LA-5061-MS
Informal Report
UC-32 and 34

ISSUED: October 1972



A Monte Carlo Sampler

by

C. J. Everett
E. D. Cashwell



A MONTE CARLO SAMPLER

by

C. J. Everett and E. D. Cashwell

ABSTRACT

Methods are given for sampling the standard probability densities arising in physical, chemical, and statistical problems, by means of machine generated "random numbers." The probability theory underlying each device is briefly indicated. The collection is intended as a reference work for Monte Carlo practice. No attempt is made to quote original sources, and no claim to priority is intended in any case.

FOREWORD

In all cases, the density to be sampled is followed by a rule (R) for choice of the variable, in terms of random numbers r_0, r_1, \dots , uniform on (0,1). A justification (J) for the rule is usually given, frequently supported by various formulas (F). The indices D, C, R provide "key words" which may help in locating a desired density, but details in this direction are omitted.

FORMULAS

$$F1. \quad \frac{d}{du} \int_{g(u)}^{h(u)} f(u,v) dv =$$

$$h'(u)f(u,h(u)) - g'(u)f(u,g(u)) + \int_{g(u)}^{h(u)} \frac{\partial}{\partial u} f(u,v) dv$$

$$F2. \quad \frac{d}{du} \int_{\{v_1 + v_2 < u, v_i > 0\}} p_1(v_1) dv_1 p_2(v_2) dv_2 = \frac{d}{du} \int_0^u p_1(v_1) dv_1 \int_0^{u-v_1} p_2(v_2) dv_2$$

$$\equiv \frac{d}{du} \int_0^u f(u, v) dv = f(u, u) + \int_0^u \frac{\partial}{\partial u} f(u, v) dv = 0 + \int_0^u p_1(v) dv p_2(u-v)$$

$$F3. \int_0^v v^{n-1} e^{-bv} dv = (n-1)! b^{-n} \left\{ 1 - e^{-bv} \sum_{i=0}^{n-1} (bv)^i / i! \right\};$$

$0 < v < \infty, b > 0, n = 1, 2, 3, \dots$

$$D_b \equiv \int_1^{\infty} v^{n-1} e^{-bv} dv = (n-1)! b^{-n} e^{-b} S_b, S_b \equiv \sum_{i=0}^{n-1} b^i / i!$$

$$F4. \Gamma(n) \equiv \int_0^{\infty} u^{n-1} e^{-u} du = 2 \int_0^{\infty} v^{2n-1} e^{-v^2} dv; n \text{ real} > 0.$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(n+1) = n\Gamma(n), 2^{n-1} \Gamma(\frac{n}{2}) \Gamma(\frac{n+1}{2}) = \Gamma(\frac{1}{2})\Gamma(n).$$

For $n = 0, 1, 2, \dots, \Gamma(n+1) = n!, 0! \equiv 1.$

$$F5. B(m, n) \equiv \int_0^1 v^{m-1} (1-v)^{n-1} dv = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$= \Gamma(m)\Gamma(n)/\Gamma(m+n); m, n \text{ real} > 0.$

$$F6. \int \prod_1^n \mu_i^{s_i-1} d\mu_i = \prod_1^n \Gamma(s_i) / \Gamma\left(1 + \sum_1^n s_i\right); s_i \text{ real} > 0.$$

$\{\sum_1^n \mu_i < 1, \mu_i > 0\}$

$$F7. \quad V(u) = \int_{\{\sum_1^n v_i \leq u, v_i > 0\}} \prod_1^n dv_i = u^n/n!,$$

$$dV/du = u^{n-1}/(n-1)! = A(u)$$

$$F8. \quad V(u) = \int_{\{(\sum_1^N v_i^2)^{1/2} \leq u, v_i > 0\}} \prod_1^N dv_i = \pi^{N/2} u^N/2^{N-1} \Gamma(N/2),$$

$$dV/du = \pi^{N/2} u^{N-1}/2^{N-1} \Gamma(N/2) = A(u)$$

Note. $V(u) = 2\pi^{N/2} u^N/\Gamma(N/2)$ is the volume of the full sphere of radius u ,
 $A(u) = 2\pi^{N/2} u^{N-1}/\Gamma(N/2)$ its area. Area of unit sphere is $A(1)$
 $= 2\pi^{N/2}/\Gamma(N/2) = \int_{\Omega} d\Omega$, where Ω is the direction vector in N -space.

$$F9. \quad \zeta(n) \equiv \sum_1^{\infty} j^{-n}; \quad n \text{ real } > 1.$$

$$\zeta_a(n) \equiv \sum_1^{\infty} (-1)^{j+1} j^{-n} = (1 - 1/2^{n-1}) \zeta(n)$$

$$\zeta_u(n) \equiv \sum_1^{\infty} (2j-1)^{-n} = (1 - 1/2^n) \zeta(n)$$

$$\zeta(2n) = (-1)^{n-1} (2\pi)^{2n} B_{2n}/2(2n)!, \quad n = 1, 2, 3, \dots$$

$$B_2 = 1/6$$

$$B_4 = -1/30$$

$$B_6 = 1/42 \quad (\text{Bernoulli numbers})$$

$$\zeta(2) = \pi^2/6$$

$$\zeta(4) = \pi^4/90$$

$$\zeta(6) = \pi^6/945$$

$$\zeta(3) \approx 1.2021\dots$$

$$F10. \quad \sum_1^n F_i = 0, \text{ where}$$

$$F_i \equiv 1/(a_1 - a_i) \cdots (a_{i-1} - a_i) \cdot (a_{i+1} - a_i) \cdots (a_n - a_i), \quad n \geq 2.$$

Proof. For $f(Z) \equiv 1/(a_1 - Z) \cdots (a_n - Z)$, one has

$$(1/2\pi i) \int_C f(Z) dZ = \sum_1^n \text{Res}(a_i), \text{ where } C \text{ is any circle of radius } R > \max |a_i|,$$

$$\text{and } -\text{Res}(a_i) = F_i, \text{ as defined. But } \left| \int_C f(Z) dZ \right| \leq 2\pi R \max_C |f(Z)| \rightarrow 0.$$

as $R \rightarrow \infty$ ($n \geq 2$)

$$F11. \quad \int_0^\infty v^{n-1} dv / (\Lambda^{-1} e^v + 1) = \sum_1^\infty (-1)^{j+1} (\Lambda^j / j^n) \int_0^\infty j^n v^{n-1} e^{-jv} dv$$

$$= \zeta_a(\Lambda, n) \Gamma(n), \text{ where } \zeta_a(\Lambda, n) \equiv \sum_1^\infty (-1)^{j+1} \Lambda^j / j^n, \quad 0 < \Lambda \leq 1, \quad n > 1.$$

$$F12. \quad \int_0^\infty v^{n-1} dv \Lambda e^{-v} / (1 - \Lambda^2 e^{-2v}) =$$

$$\sum_1^\infty (\Lambda^{2j-1} / (2j-1)^n) \int_0^\infty (2j-1)^n v^{n-1} e^{-(2j-1)v} dv = \zeta_u(\Lambda, n) \Gamma(n),$$

$$\text{where } \zeta_u(\Lambda, n) \equiv \sum_1^\infty \Lambda^{2j-1} / (2j-1)^n, \quad 0 < \Lambda \leq 1, \quad n > 1.$$

$$F13. \quad K_N(u) \equiv \int_0^{\infty} \cosh N\theta e^{-u \cosh \theta} d\theta; \quad (0, \infty), \quad N > 0$$

$$(a) \quad K_N(u) = 1/2 \int_{-\infty}^{\infty} e^{-N\theta} e^{-u \cosh \theta} d\theta \quad (\text{trivial})$$

$$(b) \quad K_N(u) = 2^{-(N+1)} u^N \int_0^{\infty} x^{-(N+1)} e^{-(x + \frac{u^2}{4x})} dx \quad (e^\theta = 2x/u)$$

$$(c) \quad K_N(u) = \frac{\Gamma(1/2) u^N}{2^N \Gamma(N + 1/2)} \int_1^{\infty} (v^2 - 1)^{N-1/2} e^{-uv} dv \quad (\text{cf. [12]})$$

$$= \frac{\Gamma(1/2) u^N}{2^N \Gamma(N + 1/2)} \int_0^1 x^{-(2N+1)} (1 - x^2)^{N-1/2} e^{-u/x} dx \quad (v = x^{-1})$$

$$(d) \quad K_N(2v^{1/2}) = 2^{-1} v^{N/2} \int_0^{\infty} x^{-(N+1)} e^{-(x + \frac{v}{x})} dx \quad (\text{from (b)})$$

$$F14. \quad \int_0^{\infty} x^{n-1} e^{-b\sqrt{x^2 + 1}} dx = \int_1^{\infty} e^{-bv} v(v^2 - 1)^{\frac{n}{2} - 1} dv$$

$$\equiv \int_1^{\infty} u dv = \frac{b}{n} \int_1^{\infty} (v^2 - 1)^{\frac{n}{2}} e^{-bv} dv$$

$$= (\Gamma(n/2)/\Gamma(\frac{1}{2})) (2/b)^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(b); \quad n, b > 0 \quad (\text{cf. F13(c)})$$

$$\text{F15. } \int_0^{\infty} x^{n-1} dx / (\Lambda^{-1} e^{a\sqrt{x^2+1}} + 1)$$

$$= \sum_1^{\infty} (-1)^{j+1} \Lambda^j \int_0^{\infty} x^{n-1} e^{-ja\sqrt{x^2+1}} dx$$

$$= 2^{\frac{n-1}{2}} (\Gamma(n/2)/\Gamma(\frac{1}{2})) \sum_1^{\infty} (-1)^{j+1} \Lambda^j K_{\frac{n+1}{2}}(ja) / (ja)^{\frac{n-1}{2}}; n, a > 0. \quad (\text{cf. F14})$$

$$\text{F16. } \int_1^{\infty} v^{n-1} \Lambda e^{-av} / (1 - \Lambda^2 e^{-2av}) = \sum_1^{\infty} \Lambda^{2j-1} \int_1^{\infty} v^{n-1} e^{-(2j-1)av} dv$$

$$= \sum_1^{\infty} \Lambda^{2j-1} D_{(2j-1)a}; D_b \text{ defined as in F3.}$$

$$\text{F17. } \int_0^{\infty} u^{n-1} K_N(u) du =$$

$$\left(\Gamma(\frac{1}{2}) / 2^N \Gamma(N + \frac{1}{2}) \right) \int_1^{\infty} (v^2 - 1)^{N-\frac{1}{2}} dv \int_0^{\infty} u^{n+N-1} e^{-uv} du$$

$$= \left(\Gamma(\frac{1}{2}) \Gamma(n+N) / 2^N \Gamma(N + \frac{1}{2}) \right) \int_1^{\infty} v^{-(n+N)} (v^2 - 1)^{N-\frac{1}{2}} dv$$

$$= \left(\Gamma(\frac{1}{2}) \Gamma(n+N) / 2^{N+1} \Gamma(N + \frac{1}{2}) \right) \int_0^1 \xi^{\frac{n-N}{2}-1} (1-\xi)^{(N+\frac{1}{2})-1} d\xi$$

$$= \left(\Gamma(\frac{1}{2}) \Gamma(n+N) / 2^{N+1} \Gamma(N + \frac{1}{2}) \right) B\left(\frac{n-N}{2}, N + \frac{1}{2}\right)$$

(F5)

$$= \Gamma(\frac{1}{2}) \Gamma(n+N) \Gamma\left(\frac{n-N}{2}\right) / 2^{N+1} \Gamma\left(\frac{n+N+1}{2}\right)$$

(F4)

$$= 2^{n-2} \Gamma\left(\frac{n-N}{2}\right) \Gamma\left(\frac{n+N}{2}\right); \quad 0 < N < n.$$

$$F18. \quad E_N(u) \equiv \int_1^\infty v^{-N} e^{-uv} dv = \int_0^1 x^{N-2} e^{-u/x} dx;$$

$0 < u < \infty, N > 0$. Cf. [1].

$$F19. \quad \int_0^\infty u^{n-1} E_N(u) du = \int_1^\infty v^{-N} dv \int_0^\infty u^{n-1} e^{-uv} dv$$

$$= \Gamma(n) \int_1^\infty v^{-(n+N)} dv = \Gamma(n) / (n+N-1); \quad N > 0, n > 0, n+N > 1.$$

D-INDEX

Discrete Densities

- | | |
|-----------------------------------|----------------------------------------------------------------|
| D1. $p(v); v = 0, 1, 2, \dots$ | Discrete, v finite or countable,
Bernoulli ($v = 0, 1$) |
| D2. $e^{-a} a^v / v!$ | Poisson |
| D3. $P\{f(v) = u\}$ | Density for <u>value</u> of a function |
| D4. $\prod_1^n p_i(v_i)$ | Vector density, independent variables |
| D5. $P\{f(v_1, \dots, v_n) = u\}$ | f-value density, independent variables |

D6.	$C_u^n p^u (1-p)^{n-u}$	Binomial, drawing with replacement
D7.	$C_{s-1}^{v-1} p^s (1-p)^{v-s}$	Negative binomial
D8.	$(1-q) q^{v-1}$	Geometric
D9.	$a^{v-1}/(1+a)^v$	Pascal
D10.	$(1+\alpha\beta)^{-1/\beta} (\alpha/1+\alpha\beta)^\mu \times$ $(1+\beta)\cdots[1+(\mu-1)\beta]/\mu!$	Polya ($\beta = 1, 1/2, 1/3, \dots$)
D11.	$(n!/n_1!\cdots n_f!) p_1^{n_1}\cdots p_f^{n_f}$	Multinomial, macrostate, particles in boxes.
D12.	$C_v^M C_{m-v}^N / C_m^{M+N}$	Hypergeometric, drawing without replacement
D13.	$C_k^n \frac{b\cdots[b+(k-1)s]\cdot c\cdots[c+(n-k-1)s]}{N\cdot(N+s)\cdots[N+(n-1)s]}$	Polya's urn
D14.	$1/N (N-1)\cdots(N-n+1)$	Random permutation
D15.	$1/C_n^N$	Random combination
D16.	$\prod_1^\infty p_i (v_i)$	Random sequences of integers

Discrete Densities

D1. $p(v); v = 0, 1, 2, \dots$

$$\mathfrak{R}. \text{ Set } v = \min \left\{ k; \sum_0^k p(v) \geq r_0 \right\}$$

D2. $p(v) = e^{-a} a^v/v! ; v = 0, 1, 2, \dots, a > 0.$

$$\mathfrak{R}1. \text{ Set } v = \min \left\{ n ; \sum_0^n a^v/v! \geq r_0 e^a \right\}$$

$$\mathfrak{R}2. \text{ Set } v = -1 + \min \left\{ n ; \prod_1^n r_i < e^{-a} \right\}$$

J2. By C2, C7, F7, F3, we have

$$P \left\{ r_1 \cdots r_n \leq e^{-a} \right\} = \int \prod_1^n dr_i \Big|_{\left\{ \prod_1^n r_i \leq e^{-a} \right\}} = \int \prod_1^n e^{-v_i} dv_i \Big|_{\left\{ \sum_1^n v_i \geq a, v_i > 0 \right\}}$$

$$= (1/(n-1)!) \int_a^\infty u^{n-1} e^{-u} du = e^{-a} \sum_0^{n-1} a^i / i! \quad . \quad \text{Cf. C22.}$$

D3. $q(u) = P\{f(v) = u\} = \sum_{\{v; f(v) = u\}} p(v)$

R. Sample $p(v)$ for v ; set $u = f(v)$.

D4. $p(v) = p(v_1, \dots, v_n) = \prod_1^n p_i(v_i)$

R. Sample each $p_i(v_i)$ for v_i ; set $v = (v_1, \dots, v_n)$

D5. $q(u) = P\{f(v_1, \dots, v_n) = u\} = \sum_{\{f(v) = u\}} p_1(v_1) \cdots p_n(v_n)$

R. Sample each $p_i(v_i)$ for v_i ; set $u = f(v_1, \dots, v_n)$.

D6. $q(u) = C_u^n (1-p)^{n-u} p^u$; $u = 0, 1, \dots, n$, $0 < p < 1$.

R. Set $u =$ number of r_1, \dots, r_n such that $r_i \leq p$.

J. For $v_i \in \{0, 1\}$, $p_i(0) \equiv 1-p$, $p_i(1) \equiv p$, $f(v) = v_1 + \dots + v_n$,

one has $P\{f(v_1, \dots, v_n) = u\} = \sum_{\{v_1 + \dots + v_n = u\}} p_1(v_1) \cdots p_n(v_n)$

$$= C_u^n (1-p)^{n-u} p^u. \quad (D5)$$

D7. $p(v) = C_{s-1}^{v-1} p^s q^{v-s}$; $v = s, s+1, \dots, 0 < p < 1, q = 1-p, s \geq 1$.

R. Set $v =$ first n for which s of the random numbers r_1, \dots, r_n are $\leq p$.

J. $p(v) = \left\{ C_{s-1}^{v-1} p^{s-1} q^{(v-1)-(s-1)} \right\}$. p is the probability of exactly s "successes" occurring for the first time at the v -th trial.

Note. $1 = p^s (1-q)^{-s} = \sum_0^{\infty} \frac{(-s)(-s-1)\dots(-s-\mu+1)}{\mu!} p^s (-q)^\mu$

$$= \sum_0^{\infty} \frac{(s+\mu-1)\dots(s+1)s}{\mu!} p^s q^\mu = \sum_0^{\infty} C_\mu^{s+\mu-1} p^s q^\mu$$

$$= \sum_s^{\infty} C_{v-s}^{v-1} p^s q^{v-s} = \sum_s^{\infty} C_{s-1}^{v-1} p^s q^{v-s}. \text{ Hence the term "negative binomial."}$$

D8. $g(v) = pq^{v-1}$; $v = 1, 2, 3, \dots, 0 < p < 1, q = 1-p$.

R 1. Set $v = \min \{n; r_n \leq p, n = 1, 2, \dots\}$.

J 1. Case $s = 1$ of D7.

R 2. Set $v = k$, where $k-1 < \ln r_1 / \ln q \leq k, k = 1, 2, \dots$

J 2. The rule follows from D1 (with $v = 1, 2, \dots$)

For, $\sum_1^k g(v) = 1-q^k$. (We have replaced r_0 by $r_1 = 1-r_0$).

D9. $g(v) = a^{v-1} / (1+a)^v$; $v = 1, 2, 3, \dots, a > 0$.

R 1. Set $v = \min \{n; r_n \leq 1/(1+a), n = 1, 2, \dots\}$

R 2. Set $v = k$, where $k-1 < \ln r_1 / \{\ln a - \ln(1+a)\} \leq k$.

J. Special case of D8, with $p = 1/(1+a)$.

$$\text{D10. } h(\mu) = (1+\alpha\beta)^{-1/\beta} \left(\frac{\alpha}{1+\alpha\beta} \right)^\mu \frac{1 \cdot (1+\beta) \cdots (1+(\mu-1)\beta)}{\mu!},$$

$$h(0) = (1+\alpha\beta)^{-1/\beta}; \mu = 0, 1, 2, \dots, \alpha > 0, \beta = 1, 1/2, 1/3, \dots$$

R. Define $s = 1/\beta$, $p = 1/(1+\alpha\beta)$. Set $\mu =$

- $s + \{\text{first } n \text{ for which } s \text{ of the random numbers } r_1, \dots, r_n \text{ are } \leq p\}$.

J. One finds that $h(\mu) = C_{s-1}^{v-1} p^s q^{v-s}$, where $v = s + \mu$.

The rule follows from D7.

$$\text{D11. } p(n_1, \dots, n_f) = (n! / n_1! \cdots n_f!) p_1^{n_1} \cdots p_f^{n_f};$$

$$n_h \geq 0, n_1 + \dots + n_f = n, p_1 + \dots + p_f = 1.$$

R. One follows the steps:

1. Put $0 \rightarrow n_1, \dots, 0 \rightarrow n_f; 1 \rightarrow t$.

2. Set $K = \min \left\{ k; \sum_1^k p_i \geq r_t \right\}$. Put $1+n_K \rightarrow n_K$.

3. If $t < n$, put $t+1 \rightarrow t$, return to (2). Otherwise exit with (n_1, \dots, n_f) .

J. If v_1, \dots, v_n are independent, each with density given by

$$v_i = 1, \dots, f \\ p_i(v_i) = p_1, \dots, p_f$$

then the vector (v_1, \dots, v_n) has density $p(v_1, \dots, v_n) =$

$p_{v_1} \dots p_{v_f}$, where $\sum_v p_{v_1} \dots p_{v_f} = (p_1 + \dots + p_f)^n = 1$. Hence, the probability,

under this density, of the class C of vectors with n_1 components 1, ..., n_f components f, is

$$\sum_{v \in C} p_{v_1} \dots p_{v_f} = \sum_{v \in C} p_1^{n_1} \dots p_f^{n_f} = (n! / n_1! \dots n_f!) p_1^{n_1} \dots p_f^{n_f},$$

and the rule follows.

Note. The density $q(u)$ for the sum $u = v_1 + \dots + v_f$ is not easily expressed (although easily sampled). Only for $f = 2$ do we have the connection with the binomial D6.

D12. $p(v) = \frac{C_v^M C_{m-v}^N}{C_m^{M+N}}$; $\max(0, m-N) \leq v \leq \min(m, M)$, $M, N \geq 1$, $1 \leq m \leq M+N$.

R. One follows the steps:

1. List the integers $1, 2, \dots, M+N$.
2. Put $0 \rightarrow v$, $M+N \rightarrow b$, $1 \rightarrow t$.
3. Set $K = \min \{k; k \geq br_t\}$, ($k = 1, 2, \dots, b$).
4. Delete K^{th} integer $\equiv a_K$ of the remaining list. If $a_K \leq M$, put $v+1 \rightarrow v$.
5. If $v = M$, or if $v < M$ and $t = m$, exit with v . Otherwise, go to (6).
6. Put $b-1 \rightarrow b$, $t+1 \rightarrow t$ and return to (3).

J. One notes that $\frac{C_v^M C_{m-v}^N}{C_m^{M+N}} = P_v^M P_{m-v}^N \cdot \frac{C_v^m}{P_m^{M+N}}$

where $P_r^n \equiv n(n-1)\dots(n-r+1)$; i.e., $p(v)$ is the chance of m drawings without replacement yielding exactly v integers $\leq M$.

D13. $p(k) = \frac{C_k^n \frac{b(b+s) \dots (b+(k-1)s) c(c+s) \dots (c+(n-k-1)s)}{N(N+s) \dots (N+(n-1)s)}}{}$;

$k = 0, 1, 2, \dots, n$, $b, c \in \{1, 2, 3, \dots\}$, $N = b+c$, $s = 0, 1, 2, \dots$

R. One follows the steps:

1. Put $0 \rightarrow k$, $1 \rightarrow t$, $b \rightarrow b'$, $N \rightarrow N'$
2. If $r_t \leq b'/N'$, put $k+1 \rightarrow k$, $b' + s \rightarrow b'$; go to (3).
If $r_t > b'/N'$, go to (3).
3. Put $N' + s \rightarrow N'$
4. If $t < N$, put $t+1 \rightarrow t$, return to (2). If $t = n$, exit with k .

J. $p(k)$ is the probability of drawing k black balls in n trials from an urn initially containing b black and c white balls ($b+c = N$), subject to the condition:

- (c) On the t -th drawing, the ball drawn is replaced, and s more balls of the same color as that drawn are added to the urn. (For $s = 0$, $p(k)$ is the binomial density).

D14. $p(\Pi) = 1/N(N-1)\dots(N-n+1)$; $1 \leq n \leq N$, $\Pi = (C_1, \dots, C_n)$ permutation of $1, \dots, N$, "taken n at a time".

R 1. Follow the steps:

1. List $1, 2, \dots, N$.
2. Put $1 \rightarrow t$, $N \rightarrow D$
3. Set $K = \min \{k; k \geq Dr_t, k = 1, 2, \dots, D\}$
4. Set $C_t = K$ -th integer of remaining list, and delete this integer from list.
5. If $t < n$, put $t+1 \rightarrow t$, $D-1 \rightarrow D$, and return to (3).
Otherwise, exit with random permutation $\Pi = (C_1, \dots, C_n)$.

R 2. (For $n = N$ only!). Follow the steps:

1. Put $1 \rightarrow A_1, \dots, N \rightarrow A_N$ (A_i storage position)
2. Put $1 \rightarrow t$, $N \rightarrow D$.
3. Set $K = \min \{k; k \geq Dr_t, k = 1, 2, \dots, D\}$
4. Interchange contents of A_D and A_K . (if $D \neq K$)
5. If $t < n$, put $t+1 \rightarrow t$, $D-1 \rightarrow D$ and return to (3)
Otherwise, exit with random permutation $\Pi = (C_1, \dots, C_N)$,
 $C_i = \underline{\text{content}}$ of A_i .

D15. $p(c) = 1/C_n^N$; $1 \leq n \leq N$, $c = \{C_1, \dots, C_n\}$ unordered set of n integers from the set $\{1, \dots, N\}$.

R 1. Follow D14, R 1, and take the set of integers $\{C_1, \dots, C_n\}$ in the permutation $\Pi = (C_1, \dots, C_n)$ there obtained.

R 2. Follow the steps:

1. Put $0 \rightarrow m$ $1 \rightarrow t$
 $n \rightarrow C$ $N \rightarrow D$
2. If $r_t \leq C/D$, put $m+1 \rightarrow m$; then set $C_m = t$; go to (3). If not go to (6).
3. If $m < n$, go to (4). If $m = n$, go to (9).
4. Put $C-1 \rightarrow C$, $D-1 \rightarrow D$; go to (5).
5. Put $t+1 \rightarrow t$; return to (2).
6. Put $D-1 \rightarrow D$; go to (7).
7. If $C < D$, go to (5). If $C = D$, go to (8).
8. Set $C_{m+1} = t+1, \dots, C_n = N$; go to (9).
9. Exit with random set $c = (C_1, \dots, C_n)$ (arranged in order of magnitude).

J. If m choices have been made through the t -th trial, ($t = 1, 2, \dots, N-1$), one accepts $t+1$ as the $m+1$ -st choice with probability $(n-m)/(N-t)$, where $N-t$ is the number of remaining candidates $t+1, \dots, N$, and $n-m$ is the number of choices still to be made from these candidates.

D16. $\prod_1^\infty p_i(v_i)$

R. Methods are given in [7, 8], based on "Poisson sequences of trials", for producing random sequences of integers of stipulated asymptotic density.

C-INDEX

Continuous Densities

- | | |
|---------------------------|-------------------------------------------------------------------|
| C1. $p(v)$ | General density, continuous on open interval, finite or infinite. |
| C2. $q(\mu)d\mu = p(v)dv$ | Change of variable. |
| C3. $\sum_1^J a_i(v)$ | Sum of positive terms, interpolated densities. |

C4.	$q(u) = \frac{d}{du} P\{f(v) \leq u\}$	Density for <u>value</u> of a function.
C5.	$\prod_1^n p_i(v_i) dv_i$	Vector density, independent variables.
C6.	$q(u) = \frac{d}{du} P\{f(v_1, \dots, v_n) \leq u\}$	f-value density, independent variables.
C7.	$q(u) = F(u) A(u)$	Case $\prod_1^n p_i(v_i) = F(f(v_1, \dots, v_n))$ of C6.
C8.	$s(u), p(u), q(u), q_1(u)$	Densities for $v_1+v_2, v_1v_2, v_2/v_1$
C9.	$p(v)$	A general device.
C10.	$1/(b-a)$	Uniform.
C11.	c_0+c_1v	Linear, disk radius
C12.	$c_0+c_1v+c_2v^2$	Quadratic, shell radius.
C13.	u^{m-1}	Power ($m>0$), sphere radius, completely degenerate gas momentum.
C14.	u^{-1}	Hyperbola
C15.	v^{-m-1}	Power ($m>0$)
C16.	$(y+1)^{-m-1}$	Pareto ($m>0$)
C17.	e^{-av}	Exponential, Laplace I, decay time, collision distance (free path $\lambda=1/a$).
C18.	$e^{-a_1u} - e^{-a_2u}$	Difference of exponentials.
C19.	$\sum_1^n F_i e^{-a_i u}$	Exponential convolute.
C20.	$\sum_1^n B_j e^{-a_j u}$	Sum of exponentials ($B_j>0$).
C21.	$B_1 e^{-a_1 u} + B_2 e^{-a_2 u}$	Hyperexponential, residence times.
C22.	$u^{n-1} e^{-u}, n = 1, 2, 3, \dots$	Γ -type, Erlangian
C23.	$v^{n-1} / (e^v - 1), n = 2, 3, \dots$	Planck type, Bose-Einstein, photons
C24.	$u^{2n-1} / (e^{u^2} - 1)$	Version of C23.
C25.	$v^{2n-1} e^{-v^2}, n = 1, 2, 3, \dots$	Gauss type, Rayleigh, Maxwell
C26.	Re^{-R^2}	Gauss type, $n = 1$.
C27.	$e^{-v^2}, (0, \infty)$	Error function
C28.	$e^{-v^2}, (-\infty, \infty)$	Normal
C29.	$u^{2n-1} e^{-u^2}, n = 1/2, 3/2, \dots$	Gauss type, Maxwell speed.

C30.	$v^{2n-1}/(e^{v^2}-1), n = 3/2, 5/2, \dots$	Planck version.
C31.	$u^{n-1}/(e^u-1), n = 3/2, 5/2, \dots$	Planck type.
C32.	$v^{n-1}e^{-v}, n = 1/2, 3/2, \dots$	Γ -type, Maxwell energy, fission spectrum.
C33.	$y^{A-1} \ln^{n-1}(1/y)$	Power-log power.
C34.	$t^{np-1}e^{-t^p}$	Γ -version
C35.	$u^{m-1}/(1+u)^{m+n}$ $v^{m-1}(1-v)^{n-1}$ $\sin^{2m-1} \theta \cos^{2n-1} \theta$ $w^{mp-1}(1-w^p)^{n-1}$	B-types, "arc sin", powers of sin, cos.
C36.	$\exp\left(-\sum_1^N v_i^2\right)$	N-normal, Maxwell velocity.
C37.	$p(\Omega), \Omega = (\omega_1, \dots, \omega_N)$	Uniform (isotropic) direction, point on unit N-sphere.
C38.	$F\left(\sqrt{\sum_1^N \xi_i^2}\right)$	Radially symmetric density.
C39.	$u^{N/2-1}e^{-u/2b}$	χ^2 density.
C40.	$x^{N-1}e^{-x^2/2b}$	χ density, Rayleigh.
C41.	$v^{N/2-1}e^{-Nv/2b}$	Mean square, χ^2/N .
C42.	$x^{N-1}e^{-Nx^2/2b}$	Root mean square, $\sqrt{\chi^2/N}$.
C43.	$1/(1+t^2/N)^{\frac{N+1}{2}}$	Student's t.
C44.	$1/(1+t^2)$	Cauchy
C45.	$F^{M/2-1}/(1+MF/N)^{\frac{M+N}{2}}$	Snedecor's F
C46.	$v^{n-1}e^{-v}/(1-\Lambda^2e^{-2v})$	Lemma for R8.
C47.	$v^{n-1}e^{-bv}, (1, \infty)$	Lemma for R9.
C48.	$v^{n-1} \Lambda e^{-av}/(1-\Lambda^2e^{-2av})$	Lemma for R 10.

- C49. $u^{n-1} E_N(u)$ Schlömilch, neutron diffusion
- C50. $u^{n-1} K_N(u)$ Bessel
- C51. $e^{-a|u-b|}$ Bilateral exponential, Laplace II.
- C52. $x^{b-1} e^{-ax^b}$ Weibull
- C53. $u^{-1} \exp \{-(\ln^2 u)/2b\}$ Log normal.
- C54. $x^{-(n+1)} e^{-a^2/2x}$ One sided stable, recurrence times.
- C55. $s(u) = \begin{cases} u; & 0 < u < 1 \\ 2-u; & 1 < u < 2 \end{cases}$ Triangular.
- C56. $\cosh \theta$ Hyperbolic
- C57. $\sinh \theta$ Hyperbolic
- C58. $e^{Bx}/(1+Bx)^2$ Kahn, approximate normal.
- C59. $y^{a_1-1} - y^{a_2-1}$ Difference of powers.
- C60. $\sum F_i y^{a_i-1}$ Power convolute.
- C61. $P(\alpha^-/\alpha)$ Klein-Nishina total cross section.
See R 7C,D and R 14 for polarized case.

Continuous Densities

C1. $p(v); (a,b)$

R. Define $P(v) = \int_a^v p(v)dv$, $P_1(v) = \int_v^b p(v)dv$. Set $v = P^{-1}(r_0)$ or $v = P_1^{-1}(r_1)$.

C2. $q(\mu)d\mu = p(v)dv$; $\mu = f(v)$ monotone

R. If preferable, sample $p(v)$ for v ; set $\mu = f(v)$.

Note that $p(v) = q(f(v)) d\mu/dv$. A similar rule applies to the n variable case, with $d\mu/dv$ replaced by the absolute Jacobian $|\det[\partial\mu_i/\partial v_j]|$ of the transformation $\mu = f(v)$.

C3. $p(v) = \sum_1^J a_j(v); (a,b), a_j(v) \geq 0, J$ finite or infinite.

R. Define $A_j = \int_a^b a_j(v)dv$. Set $K = \min \{k; \sum_1^k A_j \geq r_0\}$. Sample density $a_K(v)/A_K$ for v .

Note. This provides a quick way of sampling an interpolated density $p(v) = \alpha_1 p_1(v) + \alpha_2 p_2(v), \alpha_i > 0, \alpha_1 + \alpha_2 = 1$.

C4. $q(u) = \frac{d}{du} P\{f(v) \leq u\} = \frac{d}{du} \int_{f(v) \leq u} p(v)dv; (a,b), p(v)$ density for v ,

$P\{f(v) \leq a\} = 0$ (always assumed!)

R. Sample $p(v)$ for v ; set $u = f(v)$.

J. Since $\int_a^u q(u)du = P\{f(v) \leq u\}$, $q(u)$ is the density for the value u of the function $f(v)$.

C5. $p(v)dv = \prod_1^n p_i(v_i)dv_i$

R. Sample each $p_i(v_i)$ for v_i . Set $v = (v_1, \dots, v_n)$

C6. $q(u) = \frac{d}{du} P\{f(v_1, \dots, v_n) \leq u\} = \frac{d}{du} \int \prod_1^n p_i(v_i)dv_i; (a,b),$
 $f(v_1, \dots, v_n) \leq u$

$p_i(v_i)$ density for $v_i, i = 1, \dots, n$.

R. Sample each $p_i(v_i)$ for v_i . Set $u = f(v_1, \dots, v_n)$.

C7. $q(u) = F(u)A(u)$; (a, b) , $\prod_1^n p_i(v_i) = F(f(v_1, \dots, v_n))$, $A(u) = dV/du$,

$$V(u) = \int_{f(v) \leq u} \prod_1^n dv_i.$$

R. Sample each $p_i(v_i)$ for v_i . Set $u = f(v_1, \dots, v_n)$.

J. If v_i has density $p_i(v_i)$, and $\prod_1^n p_i(v_i) = F(f(v_1, \dots, v_n))$, where $F(u)$ is a function of one variable, and if moreover $A(u)$ is defined as above, then consideration of the one-parameter family of surfaces $f(v_1, \dots, v_n) = u$ leads to the result

$$P\{f(v) \leq u\} = \int_{f(v) \leq u} \prod_1^n p_i(v_i) \prod_1^n dv_i = \int_a^u F(u)A(u)du.$$

The rule therefore follows from C6.

C8. $s(u) = \int_0^u p_1(v_1) p_2(u-v_1)dv_1$; $(0, \infty)$, p_1, p_2 densities on $(0, \infty)$

$p(u) = \int_0^\infty p_1(v_1)v_1^{-1} p_2(v_1^{-1}u)dv_1$; $(0, \infty)$, p_1, p_2 densities on $(0, \infty)$

$q(u) = \int_0^\infty p_1(v_1)v_1 p_2(v_1 u)dv_1$; $(0, \infty)$, p_1, p_2 densities on $(0, \infty)$

$q_1(u) = \int_0^\infty p_1(v_1) v_1 p_2(v_1 u)dv_1$; $(-\infty, \infty)$, p_1, p_2 densities on $(0, \infty)$ and

$(-\infty, \infty)$, respectively.

R. In all cases, sample $p_1(v_1)$ for v_1 , $p_2(v_2)$ for v_2 . For $s(u)$, set $u = v_1 + v_2$; for $p(u)$ set $u = v_1 v_2$; for $q(u)$ or $q_1(u)$, set $u = v_2 / v_1$.

J. In each of the functions $f(v_1, v_2) = v_1 + v_2, v_1 v_2, v_2 / v_1$, one verifies that

$$\frac{d}{du} \int_{f(v) \leq u} p_1(v_1) dv_1 p_2(v_2) dv_2 \text{ has the form of the corresponding density above.}$$

Verification for $v_1 + v_2$ is given in F2.

C9. $p(v)$; (a, b)

R. Define $P(v), P_1(v)$ as in C1, and

$$f(r) = r^{-1} p(a + (b-a)r) \quad 0 < r \leq 1.$$

$$g(r) = r^{-1} p_1(b - (b-a)r)$$

(a) If $f(r)$, in particular if $p(v)$, is increasing, set

$$u = \begin{cases} r_1 & \text{if } r_2 \leq f(r_1) \\ f^{-1}(r_2) & \text{if } r_2 > f(r_1) \end{cases} \quad v = a + (b-a)u$$

(b) If $g(r)$ is increasing, in particular if $p(v)$ is decreasing, set

$$u = \begin{cases} r_1 & \text{if } r_2 \leq g(r_1) \\ g^{-1}(r_2) & \text{if } r_2 > g(r_1) \end{cases} \quad v = b - (b-a)u$$

J. For example, if $f(r)$ is increasing, and we define the function

$$F(r, s) = \begin{cases} r & \text{if } s \leq f(r) \\ f^{-1}(s) & \text{if } s > f(r) \end{cases}$$

on the unit square,

then for $p_1(r) \equiv 1 \equiv p_2(s)$, we have $\int_0^v q(u)du \equiv$

$$\int_{F(r,s) \leq v} p_1(r)p_2(s) drds = vf(v) = P(a+(b-a)v)$$

$$= \int_a^{a+(b-a)v} p(v)dv, \text{ and the rule in (a) follows from C6, C2.}$$

Note. The method is practical only if f, g are more easily invertible than P, P_1 , which is indeed true for all linear densities, and for quadratic densities on certain intervals (Cf. C11,12). For details, see [2].

C10. $p(v) = 1/(b-a); (a,b)$

R. Set $v = a+(b-a)r_0$. (C1)

C11. $p(v) = C^{-1}(c_0+c_1v); (a,b), c_1 \neq 0, C = (b-a)\{c_0+\frac{1}{2}c_1(a+b)\}$.

R. (a) If $c_1 > 0$, set $v = a + \max \left\{ (b-a)r_1, (b+a+2c_0c_1^{-1})r_2 - 2(a+c_0c_1^{-1}) \right\}$.

(b) If $c_1 < 0$, set $v = b - \max \left\{ (b-a)r_1, -(b+a+2c_0^{-1}c_1)r_2 + 2(b+c_0c_1^{-1}) \right\}$. (C9)

Note. For $p(v) = 2v/(b^2-a^2)$, set $v = a + \max \left\{ (b-a)r_1, (b+a)r_2 - 2a \right\}$. (C11)

or $v = \left\{ a^2 + (b^2 - a^2)r_0 \right\}^{1/2}$ (C1) (radius, uniform circular disk).

C12. $p(v) = C^{-1}p_1(v), p_1(v) = c_0+c_1v+c_2v^2; (a,b), c_2 \neq 0,$

$C = (b-a) \left\{ c_0 + \frac{1}{2}c_1(b+a) + \frac{1}{3}c_2(b^2+ab+a^2) \right\}$, for certain intervals.

R. (a) If $c_2 > 0$, $p_1'(a) \geq 0$; or if $c_2 < 0$, $a < -\frac{1}{2}c_1c_2^{-1}$, $b \leq -\frac{1}{2}(a + \frac{3}{2}c_1c_2^{-1})$

$$\text{set } v = \begin{cases} a+(b-a)r_1 & \text{if } r_2 \leq f(r_1) \\ a+\lambda(r_2) & \text{if } r_2 > f(r_1) \end{cases}$$

where $f(r) \equiv (b-a)c^{-1} \left\{ p_1(a) + \frac{1}{2} p_1'(a) (b-a)r + \frac{1}{3} c_2(b-a)^2 r^2 \right\}$,

$$\lambda(s) \equiv \frac{1}{2} \left\{ -\alpha + \text{sgn } c_2 \sqrt{\alpha^2 + 12c_2^{-1} \left(\frac{c}{b-a} s - p_1(a) \right)} \right\}$$

and $\alpha \equiv 3 \left(a + \frac{1}{2} c_1 c_2^{-1} \right)$

(b) If $c_2 > 0$, $p_1'(b) \leq 0$; or if $c_2 < 0$, $b > -\frac{1}{2}c_1c_2^{-1}$, $a \geq -\frac{1}{2}(b + \frac{3}{2}c_1c_2^{-1})$,

$$\text{set } v = \begin{cases} b-(b-a)r_1 & \text{if } r_2 \leq g(r_1) \\ b-\mu(r_2) & \text{if } r_2 > g(r_1) \end{cases}$$

where $g(r) \equiv (b-a)c^{-1} \left\{ p_1(b) - \frac{1}{2} p_1'(b)(b-a)r + \frac{1}{3} c_2(b-a)^2 r^2 \right\}$,

$$\mu(s) \equiv \frac{1}{2} \left\{ \beta + \text{sgn } c_2 \sqrt{\beta^2 + 12c_2^{-1} \left(\frac{c}{b-a} s - p_1(b) \right)} \right\}$$

and $\beta \equiv 3 \left(b + \frac{1}{2} c_1 c_2^{-1} \right)$. (C9)

Note. The method, when applicable, is indicated if C1 involves a difficult cubic. For $p(v) = 3v^2/(b^3-a^3)$ (radius, uniform spherical shell), C1 sets

$$v = \left\{ a^3 + (b^3 - a^3)r_0 \right\}^{1/3}, \text{ whereas C12 sets}$$

$$v = \begin{cases} a+(b-a)r_1 & \text{if } r_2 \leq f(r_1) \\ a+\lambda(r_2) & \text{if } r_2 > f(r_1) \end{cases}, \text{ where}$$

$$f(r) = \{3a^2+3a(b-a)r + (b-a)^2 r^2\}/(b^2+ba+a^2) \equiv \bar{A}+r(\bar{B}+r\bar{C})$$

$$\lambda(s) = \frac{1}{2} \left\{ -3a + \sqrt{4(b^2+ba+a^2)s-3a^2} \right\} \equiv \bar{D} + \sqrt{\bar{E}s-\bar{F}}; \bar{A}, \dots, \bar{F} \text{ stored.}$$

Machine times should be compared. For $a = 0$, see also C13, $m = 3$.

C13. $q(u) = mb^{-m}u^{m-1}; (0, b), m = k/\ell, k, \ell \in \{1, 2, 3, \dots\}$

R. Set $u = b (\max \{r_1, \dots, r_k\})^\ell$

J. For $\ell = 1, m = k$, let $p_i(v_i) \equiv 1, i = 1, \dots, k$, and $f(v) =$

$$b \max \{v_1, \dots, v_k\}. \text{ Then } \int_{f(v) \leq u} \prod_1^k p_i(v_i) dv_i = b^{-k} u^k = \int_0^u kb^{-k} u^{k-1} du,$$

and the rule follows from C6. For $\ell \geq 2$, and $u = b^{1-\ell} v^\ell$, note that $q(u)du = kb^{-k} v^{k-1} dv$, which reduces the second case to the first.

C13A. $q(u) = C^{-1}u^{m-1}; (a, b), m \text{ real } > 0, C = (b^m - a^m)/m, a \geq 0.$

R. Set $u = \{a^m + (b^m - a^m)r_0\}^{1/m}. \quad (C1)$

Note. For $m = 1/\ell, a = 0$, this gives the rule of C13.

C14. $q(u) = C^{-1}u^{-1}; 0 < a < u < b < \infty, C = \ln(b/a)$

R. Set $u = ae^{Cr_0} \quad (C1)$

C15. $p(v) = m\beta^m v^{-m-1}$; $0 < \beta < v < \infty$, $m = 1/2, 1, 3/2, 2, \dots$

R. For $m = 1, 2, 3, \dots$, set $v = \beta / \max \{r_1, \dots, r_m\}$.

For $m = 1/2, 3/2, \dots$, set $v = \beta / (\max \{r_1, \dots, r_{2m}\})^2$.

J. For $v = u^{-1}$, one finds $p(v)dv = m(1/\beta)^{-m} u^{m-1}(-du)$, $0 < u < 1/\beta$.
and the rule follows from C13.

C15A. $p(v) = C^{-1} v^{-m-1}$; $0 < \beta < u < \infty$, $m \text{ real } > 0$, $C = (\beta^{-m} - \alpha^{-m})/m$.

R. Set $v = 1 / \{\alpha^{-m} + (\beta^{-m} - \alpha^{-m})r_0\}^{1/m}$. (C1)

C16. $q(y) = m(y+1)^{-(m+1)}$; $(0, \infty)$, $m \text{ real } > 0$.

R. Set $y = v-1$, where v is obtained from C15 or C15A with $\beta = 1$.

J. For $y = v-1$, one has $q(y)dy = mv^{-m-1}dv$, $1 < y < \infty$. (C1)

C17. $p(v) = ae^{-av}$; $(0, \infty)$, $a > 0$.

R. Set $v = -a^{-1} \ln r_0$ (C1)

C18. $q(u) = \frac{a_1 a_2}{a_2 - a_1} (e^{-a_1 u} - e^{-a_2 u})$; $(0, \infty)$, $0 < a_1 < a_2 \text{ real}$.

R. Set $u = -(a_1^{-1} \ln r_1 + a_2^{-1} \ln r_2)$

J. For $p_1(v_1) = a_1 e^{-a_1 v_1}$, $p_2(v_2) = a_2 e^{-a_2 v_2}$ on $(0, \infty)$, one

finds $\int_0^u p_1(v_1) p_2(u-v_1) dv_1 = q(u)$ above. The rule follows from C8(s), C17.

$$C19. \quad q(u) = \sum_1^n \frac{(a_1 \dots a_n)}{(a_1 - a_i) \dots (a_{i-1} - a_i) \cdot (a_{i+1} - a_i) \dots (a_n - a_i)} e^{-a_i u}; \quad (0, \infty),$$

$a_i > 0$, distinct, $n \geq 2$.

R. Set $u = -\sum_1^n a_i^{-1} \ln r_i$.

J. The relation $q(u) = \frac{d}{du} \int_{\sum_1^n v_i \leq u} \prod_1^n a_i e^{-a_i v_i} dv_i$ may be proved by induction,

using C18 as a basis. If F_i^n denotes the above coefficient of $e^{-a_i u}$, the induction step requires that (C8)

$$\int_0^u \sum_1^n F_i^n e^{-a_i v} dv a_{n+1} e^{-a_{n+1}(u-v)} = \sum_1^{n+1} F_i^{n+1} e^{-a_i u}. \quad \text{This result follows at}$$

once from an identity proved in F10.

$$C20. \quad p(u) = \sum_1^n B_j e^{-a_j u}; \quad (0, \infty), \quad B_j > 0.$$

R. Set $K = \min \left\{ k; \sum_1^k (B_j/a_j) \geq r_0 \right\}$ and $u = -a_K^{-1} \ln r_1$. (C3, C17).

$$C21. \quad p(u) = (2a^2/\tau) e^{-2au/\tau} + 2((1-a)^2/\tau) e^{-2(1-a)u/\tau}; \quad (0, \infty), \quad \tau > 0, \quad 0 < a \leq 1/2.$$

R. Define $B_1 = 2a^2/\tau$, $a_1 = 2a/\tau$, $B_2 = 2(1-a)^2/\tau$, $a_2 = 2(1-a)/\tau$,
whence $B_1/a_1 = a$, $B_2/a_2 = (1-a)$. Set K (= 1 or 2) and u as in C20.

$$C22. \quad q(u) = u^{n-1} e^{-u} / \Gamma(n); \quad (0, \infty), \quad n = 1, 2, 3, \dots$$

R. Set $u = -\ln \prod_1^n r_i$.

J. For $p_i(v_i) = e^{-v_i}$, $(0, \infty)$, and $f(v) = \sum_1^n v_i$, we have

$$\frac{d}{du} \int_{f(v) \leq u} \prod_1^n p_i(v_i) \prod_1^n dv_i = \frac{d}{du} \int_{f(v) \leq u} e^{-f(v)} \prod_1^n dv_i = \frac{d}{du} \int_0^u e^{-u} A(u) du$$

$$= e^{-u} A(u) = e^{-u} u^{n-1} / (n-1)! = q(u) \text{ where } A(u) = dV/du$$

and $V = \int_{f(v) \leq u} \prod_1^n dv_i = u^n / n!$ as in F7. The rule follows from C7, C17.

C23. $p(v) = v^{n-1} / (e^v - 1) \zeta(n) \Gamma(n)$; $(0, \infty)$, $n = 2, 3, 4, \dots$

R. Set $K = \min \left\{ k; \sum_1^k j^{-n} \geq r_0 \zeta(n) \right\}$, $v = -K^{-1} \ln \prod_1^n r_i$.

J. Since $p(v)dv = \sum_1^\infty (j^{-n} / \zeta(n)) \left\{ (jv)^{n-1} e^{-jv} d(jv) / \Gamma(n) \right\}$, the rule follows from C3, C22, C2. (Noted for $n = 4$ by C. Barnett, E. Canfield of LRL.)

C24. $p(u) = 2u^{2n-1} / (e^{u^2} - 1) \zeta(n) \Gamma(n)$; $(0, \infty)$, $n = 2, 3, 4, \dots$

R. Set v as in C23, and $u = \sqrt{v}$. (C2)

C25. $p(v) = 2v^{2n-1} e^{-v^2} / \Gamma(n)$; $(0, \infty)$, $n = 1, 2, 3, \dots$

R. Set $v = \{-\ln \prod_1^n r_i\}^{1/2}$ (C2, C22)

C26. $p(R) = 2Re^{-R^2}$; $(0, \infty)$

R. Set $R = \{-\ln r\}^{1/2}$ (C25)

C27. $p(v_1) = 2e^{-v_1^2/\sqrt{\pi}}; (0, \infty)$ (See also R12)

R 1. Set $v_1 = R \cos \theta, v_2 = R \sin \theta$, where $R = (-\ln r_0)^{1/2}, \theta = \frac{\pi}{2} r_1$.
(Two independent samples v_1, v_2 are obtained.)

J1. Under the indicated transformation,

$$\frac{2}{\sqrt{\pi}} e^{-v_1^2} dv_1 \cdot \frac{2}{\sqrt{\pi}} e^{-v_2^2} dv_2 = 2R e^{-R^2} dR \cdot \frac{2}{\pi} d\theta. \text{ The rule follows from C2, C26.}$$

R 2. Obtain $S = r_1^2 + r_2^2 \leq 1$ as in R1. Set

$$v_1 = \{(-\ln r_0)/S\}^{1/2} r_1, v_2 = \{(-\ln r_0)/S\}^{1/2} r_2. \text{ (Two samples)}$$

R 3. Obtain $S = r_1^2 + r_2^2 \leq 1$ as in R1. Set

$$v_1 = \{(-\ln S)/S\}^{1/2} r_1, v_2 = \{(-\ln S)/S\}^{1/2} r_2. \text{ (Two samples)}$$

J3. Under the transformation $(v_1, v_2) \leftrightarrow (\rho, \theta)$:

$$v_1 = R \cos \theta, v_2 = R \sin \theta, R \equiv \{-2 \ln \rho\}^{1/2},$$

one finds $\frac{2}{\sqrt{\pi}} e^{-v_1^2} dv_1 \frac{2}{\sqrt{\pi}} e^{-v_2^2} dv_2 = \frac{4}{\pi} \rho d\rho d\theta$. To sample the latter is to

sample the unit disk in quadrant I, uniformly in area, and this is just what R1 does. The method avoids the additional r_0 of R 2. (Box, Muller, Marsaglia.)

C28. $p_0(v_1) = e^{-v_1^2/\sqrt{\pi}}; (-\infty, \infty)$ (See also R12)

R 1. Set $v_1 = R \cos \theta, v_2 = R \sin \theta$, where $R = (-\ln r_0)^{1/2}, \theta = 2\pi r_1$.

R 2. Obtain $S \leq 1, x, y$ as in R3, R 3. Set

$$v_1 = \{(-\ln r_0)/S\}^{1/2} x, \quad v_2 = \{(-\ln r_0)/S\}^{1/2} y.$$

R 3. Obtain $S \leq 1$, x , y as in R3, R 3. Set

$$v_1 = \{(-\ln S)/S\}^{1/2} x, \quad v_2 = \{(-\ln S)/S\}^{1/2} y.$$

J. Cf. C27.

Note. Two independent samples v_1 , v_2 are obtained in each rule.

C29. $p(u) = 2u^{2n-1} e^{-u^2}/\Gamma(n)$; $(0, \infty)$, $n = 1/2, 3/2, 5/2, \dots$

R. Define $h = n-1/2$ ($h = 0, 1, 2, \dots$). Sample $2e^{-\tau^2}/\sqrt{\pi}$ for $(\tau)^2$ by any of the rules in C27. (Two samples. Save one. Avoid squaring square roots!). Set

$$u = \{-\ln \prod_1^h r_i + (\tau)^2\}^{1/2}$$

J. For $p_i(v_i) = 2e^{-v_i^2}/\sqrt{\pi}$ on $(0, \infty)$, $i = 1, \dots, 2n$, and

$$f(v) = \left\{ \sum_1^{2n} v_i^2 \right\}^{1/2}, \quad \text{one has } \prod_1^{2n} p_i(v_i) = 2^{2n} e^{-f^2(v)}/\pi^n,$$

$$A = \frac{d}{du} \int_{f(v) \leq u} \prod_1^{2n} dv_i = \pi^n u^{2n-1}/2^{2n-1} \Gamma(n), \quad (F8), \quad \text{and hence the rule follows}$$

from C7, C27.

C30. $p(v) = 2v^{2n-1}/(e^{v^2}-1)\zeta(n)\Gamma(n)$; $(0, \infty)$, $n = 3/2, 5/2, \dots$

R. Set $K = \min \left\{ k; \sum_1^k j^{-n} \geq r_0 \zeta(n) \right\}$, $h = n-1/2$.

Sample $2e^{-\tau^2}/\sqrt{\pi}$ for $(\tau)^2$ by C27 (two samples, save one. Avoid squaring square roots.) Set

$$v = \{K^{-1} [1 - \lambda n \prod_1^h r_i + (\tau)^2]\}^{1/2}. \quad (C3, C29. \text{ Cf. C23})$$

C31. $p(u) = u^{n-1} / (e^u - 1) \zeta(n) \Gamma(n); (0, \infty), n = 3/2, 5/2, \dots$

R. Obtain $(v)^2$ as in C30. (Avoid squaring the square root!) Set $u = (v)^2$.

C32. $p(v) = v^{n-1} e^{-v} / \Gamma(n); (0, \infty), n = 1/2, 3/2, 5/2, \dots$

R. Obtain $(u)^2$ as in C29. (Avoid squaring the square root.)
Set $v = (u)^2$. (C2, C29).

C33. $g(y) = A^n y^{A-1} \lambda n^{n-1} (1/y) / \Gamma(n); (0, 1), A > 0, n = 1/2, 1, 3/2, 2, \dots$

R. Set $y = e^{-x/A}$, where x is the u of C22 or the v of C32.

J. $g(y)dy = x^{n-1} e^{-x} (-dx) / \Gamma(n). \quad (C2)$

C34. $g(t) = p t^{np-1} e^{-t^p} / \Gamma(n); (0, \infty), p \text{ real } > 0, n = 1/2, 1, 3/2, 2, \dots$

R. Sample $x^{n-1} e^{-x} / \Gamma(n)$ for x on $(0, \infty)$ by C22 or C32.
Set $t = x^{1/p}$. (C2) For $p = 2$, cf. C25, C29.

C35. $b(u) = u^{m-1} / (1+u)^{m+n} \cdot B(m, n); (0, \infty)$

$p(v) = v^{m-1} (1-v)^{n-1} / B(m, n); (0, 1)$

$q(\theta) = 2 \sin^{2m-1} \theta \cos^{2n-1} \theta / B(m, n); (0, \pi/2)$

$C(w) = p w^{mp-1} (1-w^p)^{n-1} / B(m, n); (0, 1), p \text{ real } > 0$

$m, n \in \{1/2, 1, 3/2, 2, \dots\}$ in all.

R. Sample $p_1(v_1) = v_1^{n-1} e^{-v_1}/\Gamma(n)$, $p_2(v_2) = v_2^{m-1} e^{-v_2}/\Gamma(m)$

for v_1, v_2 on $(0, \infty)$ by C22 and/or C32. Set $u = v_2/v_1$, $v = u/(1+u)$,

$\theta = \arcsin \sqrt{v}$, $w = v^{1/p}$. (The densities are equivalent under the last 3 substitutions.)

J. The rule follows from C8(q), since

$$\int_0^\infty p_1(v_1) v_1 p_2(uv_1) dv_1 = b(u) \text{ above. Note. The same rule results from the}$$

$$\text{equivalence } \frac{2}{\Gamma(n)} \xi^{2n-1} e^{-\xi^2} d\xi \cdot \frac{2}{\Gamma(m)} \eta^{2m-1} e^{-\eta^2} d\eta =$$

$$\frac{2}{\Gamma(m+n)} \rho^{2(m+n)-1} e^{-\rho^2} d\rho \cdot \frac{2}{B(m,n)} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta.$$

Note. For $n = 1$, $m = 1/2, 1, 3/2, 2, \dots$, see C13.

For $n = 1/2$, observe that $q(\theta) = 2\Gamma(m + \frac{1}{2}) \sin^{2m-1} \theta / \sqrt{\pi} \Gamma(m)$,

$m = 1/2, 1, 3/2, 2, \dots$. For $n = 1/2 = m$, set $\theta = \frac{\pi}{2} r_0$, $v = \sin^2 \theta$, etc.

C36. $p(v_1, \dots, v_N) = \pi^{-N/2} \exp - \sum_1^N v_i^2; \quad -\infty < v_i < \infty, N = 1, 2, \dots$

R. For $N = 2h$, obtain $(v_1, v_2), \dots, (v_{2h-1}, v_{2h})$ from C28.

For $N = 2h+1$, obtain also (v_{2h+1}, v_{2h+2}) from C28 (save v_{2h+2}). (C5)

Note. The density of the value u of the function

$$f(v) = \left(\sum_1^N v_i^2 \right)^{1/2} \text{ is } 2u^{N-1} e^{-u^2} / \Gamma(N/2) \quad (\text{Cf. C40}).$$

C37. $p(\Omega) = \Gamma(N/2) / 2\pi^{N/2}$

R. Set v_1, \dots, v_N as in C36, and $\Omega = (\omega_1, \dots, \omega_N)$,

where $\omega_i = v_i / \rho$, $\rho = \left(\sum_1^N v_i^2 \right)^{1/2}$.

Note 1. In C28 (the source of the v_i), observe that $v_1^2 + v_2^2 = -\ln r_0$ or $-\ln S$; this saves time in computing ρ .

Note 2. The rule determines a uniformly distributed direction Ω in N -space; equivalently, a point on the unit sphere $|\Omega| = 1$ (F8).

See also R3, 4, 5.

C38. $p(\xi_1, \dots, \xi_N) = F(u)$; $u = \left(\sum_1^N \xi_i^2 \right)^{1/2}$, $-\infty < \xi_i < \infty$.

R. Sample $q(u) = 2\pi^{N/2} u^{N-1} F(u) / \Gamma(N/2)$ for u on $(0, \infty)$.
Sample unit sphere $|\Omega| = 1$ for Ω by C37. Set $\xi_i = u\omega_i$.

J.
$$\int_{\left(\sum_1^N \xi_i^2 \right)^{1/2} \leq u} p(\xi_1, \dots, \xi_N) \prod_1^N d\xi_i = \int_0^u F(u) A(u) du \quad (\text{F8, C7})$$

Note. For $F(u)$ normal, use C36.

C39. $q(u) = u^{\frac{N}{2} - 1} e^{-u/2b} / (2b)^{N/2} \Gamma(N/2)$; $(0, \infty)$, $b > 0$, $N = 1, 2, 3, \dots$

R. Sample $w^{N/2-1} e^{-w} / \Gamma(N/2)$ for w on $(0, \infty)$ by C22 or C32.

Set $u = 2bw$. (C2).

Note. $q(u)$ is the density of the value of $u = \chi^2 = \sum_1^N v_i^2$, defined in statistics as $\frac{d}{du} \int_{\sum_1^N v_i^2 \leq u} \prod_1^N \frac{1}{\sqrt{2\pi b}} e^{-v_i^2/2b} dv_i$, $-\infty < v_i < \infty$, which may be

evaluated using F8, C7.

C40. $p(x) = 2x^{N-1} e^{-x^2/2b} / (2b)^{N/2} \Gamma(N/2)$; $(0, \infty)$, $b > 0$, $N = 1, 2, 3, \dots$

R. Sample $w^{N/2-1} e^{-w} / \Gamma(N/2)$ by C22 or C32. Set $\chi = \sqrt{2bw}$. (C2)

Note. $p(x)$ is the density for $\chi = \sqrt{\sum_1^N v_i^2}$.

C41. $q(v) = N^{N/2} v^{N/2-1} e^{-Nv/2b} / (2b)^{N/2} \Gamma(N/2)$; $(0, \infty)$, $b > 0$, $N = 1, 2, 3, \dots$

R. Sample $w^{N/2-1} e^{-w} / \Gamma(N/2)$ by C22 or C32. Set $v = 2bw/N$ (C2)

Note. $q(v)$ is the density of the mean square $\sum_1^N v_i^2 / N = \chi^2 / N$.

C42. $p(x) = 2N^{N/2} x^{N-1} e^{-Nx^2/2b} / (2b)^{N/2} \Gamma(N/2)$; $(0, \infty)$, $b > 0$, $N = 1, 2, 3, \dots$

R. Sample $w^{N/2-1} e^{-w} / \Gamma(N/2)$ by C22 or C32. Set $x = \sqrt{2bw/N}$ (C2)

Note. $p(x)$ is the density of the "root mean square" $\sqrt{\sum_1^N v_i^2 / N}$
 $= \sqrt{\chi^2 / N} = \chi / \sqrt{N}$.

C43. $q_1(t) = \Gamma(\frac{N+1}{2}) / \sqrt{N\pi} \Gamma(N/2) \left(1 + \frac{t^2}{N}\right)^{\frac{N+1}{2}}$; $(-\infty, \infty)$, $N = 1, 2, 3, \dots$

R. Sample $w^{N/2-1} e^{-w}/\Gamma(N/2)$ for w on $(0, \infty)$ by C22 or C32;

set $x = (w/N)^{1/2}$. Sample $e^{-y^2}/\sqrt{\pi}$ for y on $(-\infty, \infty)$ by C28. Set $t = y/x$.

J. $q_1(t)$ is the density for the function $y/\left\{N^{-1} \sum_1^N v_i^2\right\}^{1/2}$ where the v_i , y all have density $e^{-\xi^2/2b}/\sqrt{2\pi b}$ on $(-\infty, \infty)$. For, using the density

$p_0(y) = e^{-y^2/2b}/\sqrt{2\pi b}$ for y and the $p(x)$ of C42

for $x = \sqrt{N^{-1} \sum_1^N v_i^2}$, one finds that $\int_0^\infty p(x) x p_0(tx) dx = q_1(t)$ above. The result being independent of b , the rule follows from C8, C28, C42 (with $2b = 1$).

C44. $q(t) = 1/\pi (1+t^2)$; $(-\infty, \infty)$

R. Set $t = \tan \frac{\pi}{2} (2r_0 - 1)$, or $t = y/x$ as in R2. (C1)

Note. Compare with C43 ($N = 1$).

C45. $q(F) = (M/N)^{M/2} F^{\frac{M-2}{2}} / (1 + \frac{M}{N} F)^{\frac{M+N}{2}} B(M/2, N/2)$; $(0, \infty)$, $M, N \in \{1, 2, 3, \dots\}$

R. Define $m = M/2$, $n = N/2$. Sample $v_1^{n-1} e^{-v_1}/\Gamma(n)$, $v_2^{m-1} e^{-v_2}/\Gamma(m)$

for v_1, v_2 on $(0, \infty)$ by C22 and/or C32. Set $F = Nv_2/Mv_1$.

J. For $F = Nu/M$, one finds $q(F)dF = b(u)du$ as in C35.

Note. $q(F)$ is the density for the function $M^{-1} \sum_1^M \mu_i^2 / N^{-1} \sum_1^N v_j^2$, where all μ_i, v_j have density $e^{-\xi^2/2b}/\sqrt{2\pi b}$ on $(-\infty, \infty)$. Using the densities

$q_M(\mu), q_N(v)$ of C41 for numerator and denominator, it is easy to verify

that $\int_0^\infty q_N(v) v q_M(Fv) dv = q(F)$ above, independent of b . Cf. C8.

C46. $q(v) = C^{-1} v^{n-1} \Lambda e^{-v} / (1 - \Lambda^2 e^{-2v})$; $(0, \infty)$, $0 < \Lambda \leq 1$, $n = 3/2, 2, 5/2, 3, \dots$

$$C = \zeta_u(\Lambda, n) \Gamma(n), \quad \zeta_u(\Lambda, n) \equiv \sum_1^\infty \Lambda^{2j-1} / (2j-1)^n \quad (F12)$$

R. Set $K = \min \{k; \sum_1^k \Lambda^{2j-1} / (2j-1)^n \geq r_0 \zeta_u(\Lambda, n)\}$. Sample $x^{n-1} e^{-x} / \Gamma(n)$ for x on $(0, \infty)$ by C22 or C32. Set $v = x / (2K-1)$.

J. One can write $q(v) = \sum_1^\infty \{\Lambda^{2j-1} / (2j-1)^n \zeta_u(\Lambda, n)\} \cdot$

$(2j-1)^n v^{n-1} e^{-(2j-1)v} dv / \Gamma(n)$ and apply C3, C2.

C47. $q(v) = D_b^{-1} v^{n-1} e^{-bv}$; $(1, \infty)$, $b > 0$, $n = 1, 2, 3, \dots$

$$D_b = (n-1)! b^{-n} e^{-b} S_b, \quad S_b = \sum_0^{n-1} b^i / i! \quad (F3)$$

R. Set $L = \min \{l; \sum_0^l b^i / i! \geq r_0 S_b\}$ ($0 \leq L \leq n-1$),

and $v = 1 - b^{-1} \ln \prod_1^{n-L} r_i$

J. Under the transformation $v = 1 + b^{-1}u$, one finds $q(v)dv = p(u)du$

$$\equiv D_b^{-1} b^{-n} e^{-b} (u+b)^{n-1} e^{-u} du = \sum_0^{n-1} \{S_b^{-1} b^i / i!\} u^{n-i-1} e^{-u} du / (n-i-1)!, \quad 0 < u < \infty,$$

and the rule follows from C3, C22.

Note. The partial sums of S_b should be stored.

C48. $\bar{q}(v) = D^{-1} v^{n-1} \Lambda e^{-av} / (1 - \Lambda^2 e^{-2av})$; $(1, \infty)$, $0 < \Lambda \leq 1$, $n = 2, 3, \dots$,

$$D = \sum_1^\infty \Lambda^{2j-1} D_{(2j-1)a}, \quad D_b \equiv (n-1)! b^{-n} e^{-b} S_b, \quad S_b = \sum_0^{n-1} b^i / i! \quad (F16)$$

R. Compute and store partial sums S_k of

$$S = \sum_1^{\infty} \Lambda^{2j-1} (2j-1)^{-n} e^{-2ja} S_{(2j-1)a}, \text{ where } S_b \text{ is defined above.}$$

Set $K = \min \{k; S_k \geq S_r\}$. Use C47, with $b \equiv (2K-1)a$, to obtain v on $(1, \infty)$.

C49. $q(u) = C^{-1} u^{n-1} E_N(u); (0, \infty), N \geq 0, n+N > 1, n = 1/2, 1, 3/2, 2, \dots,$

$$E_N(u) = \int_1^{\infty} v^{-N} e^{-uv} dv, C = \Gamma(n)/(n+N-1). \quad (\text{F18, 19}).$$

R. Sample $p_1(x) = (n+N-1)x^{n+N-2}$ for x on $(0,1)$ by C13, and $p_2(y) = y^{n-1} e^{-y}/\Gamma(n)$ for y on $(0, \infty)$ by C22 or C32. Set $u = xy$.

$$\begin{aligned} \text{J. } \frac{d}{du} \int_{xy \leq u} p_1(x) dx p_2(y) dy &= \frac{d}{du} \int_0^1 p_1(x) dx \int_0^{u/x} p_2(y) dy \\ &= \int_0^1 p_1(x) dx x^{-1} p_2(u/x) = C^{-1} u^{n-1} \int_0^1 x^{N-2} e^{-u/x} dx = q(u) \quad (\text{F18}). \end{aligned}$$

The rule follows from C6. (Cf. C8p)

Note. For $N = 0, n = 3/2, 2, 5/2, \dots$, sample $q(u) = u^{n-2} e^{-u}/\Gamma(n-1)$ for u on $(0, \infty)$ by C22 or C32.

C50. $q(u) = C^{-1} u^{n-1} K_N(u); (0, \infty), N = 0, 1/2, 1, 3/2, 2, \dots,$

$$n-N = 1, 2, 3, \dots, K_N(u) = \int_0^{\infty} \cosh N\theta e^{-u \cosh \theta} d\theta,$$

$$C = 2^{n-2} \Gamma\left(\frac{n-N}{2}\right) \Gamma\left(\frac{n+N}{2}\right). \quad (\text{F17})$$

R 1. Define $H = (n-N)/2, J = (n+N)/2$. Sample $p_1(x) = x^{H-1} e^{-x}/\Gamma(H)$

and $p_2(x) = x^{J-1} e^{-x}/\Gamma(J)$ for x, y on $(0, \infty)$ by C22 and/or C32.

Set $u = 2(xy)^{1/2}$.

J1. Transforming by $u = 2v^{1/2}$, we have $q(u)du = C^{-1}u^{n-1} K_N(u) du$
 $= C^{-1}2^{n-1} v^{\frac{n}{2}-1} K_N(2v^{1/2}) = \int_0^\infty p_1(x)x^{-1} p_2(x^{-1}v)dx$ (cf. F13(d)), and the rule follows from C2, C8p.

Note. The Legendre relation $2^{n-1} \Gamma(\frac{n}{2}) \Gamma(\frac{n+1}{2}) = \Gamma(\frac{1}{2}) \Gamma(n)$ may be regarded as a consequence of the above reduction!

R 2. Define $H = (n-N)/2$, $K = N + \frac{1}{2}$. Sample $p(\xi) = \xi^{H-1}(1-\xi)^{K-1}/B(H,K)$

for ξ on $(0,1)$ by C35, and $p_2(y) = y^{n+N-1} e^{-y}/\Gamma(n+N)$ for y on $(0, \infty)$ by C22.

Set $u = y\xi^{1/2}$.

J2. Define $p_1(x) = \frac{2}{B(H,K)} x^{n-N-1} (1-x^2)^{N-1/2}$. Then by F13, we may write

$$q(u) = \int_0^1 p_1(x)x^{-1} p_2(x^{-1}u)dx = \frac{d}{du} \int_{xy \leq u} p_1(x)dx p_2(y)dy. \text{ Hence, we can sample}$$

$p_1(x)$ for x on $(0,1)$ and $p_2(y)$ on $(0, \infty)$, and set $u = xy$, by C8p. But for

$x = \xi^{1/2}$, we have $p_1(x)dx = p(\xi)d\xi$, as defined above; so we may sample $p(\xi)$

for ξ on $(0,1)$ and set $x = \xi^{1/2}$. The rule follows. (Noted by Kalos [11]

for $n = N+2$.)

C51. $q(u) = (a/2) \exp-a|u-b|$; $(-\infty, \infty)$, $a > 0$, $-\infty < b < \infty$.

R. Set $u = b \pm a^{-1} \ln r_0$, the signs being equally likely. (C2, C17)

C52. $f(x) = abx^{b-1} e^{-ax^b}; (0, \infty), a, b > 0.$

R. Set $x = \exp\{b^{-1} \ln(-a^{-1} \ln r_0)\}$. (C2, C17)

C53. $q(u) = u^{-1} \exp\{-(\ln^2 u)/2b\}/(2\pi b)^{1/2}; (0, \infty), b > 0.$

R. Sample $e^{-v^2}/\sqrt{\pi}$ for v on $(-\infty, \infty)$ by C28. Set $u = e^{v\sqrt{2b}}$.

J. For $u = e^{v\sqrt{2b}}$, one has $q(u)du = e^{-v^2} dv/\sqrt{\pi}$. (C2)

Note. Since $q(u) = \frac{d}{du} \int_{e^x \leq u} e^{-x^2/2b} dx / (2\pi b)^{1/2}$, $q(u) = \frac{d}{du} P\{e^x \leq u\}$

under the density $e^{-x^2/2b}/(2\pi b)^{1/2}$.

C54. $f(x) = a^{2n} x^{-(n+1)} e^{-a^2/2x} / 2^n \Gamma(n); (0, \infty), a > 0, n = 1/2, 1, 3/2, 2, \dots$

R. Sample $y^{n-1} e^{-y}/\Gamma(n)$ for y on $(0, \infty)$ by C22 or C32. Set $x = a^2/2y$. (C2)

C55. $s(u) = \begin{cases} u; & 0 \leq u \leq 1 \\ 2-u; & 1 \leq u \leq 2 \end{cases}$

R. Set $u = r_1 + r_2$.

J. For $p_1(v_1) \equiv 1 \equiv p_2(v_2)$, v_i on $(0, 1)$, it is geometrically obvious that

$$P\{v_1 + v_2 \leq u\} = \begin{cases} u^2/2; & 0 \leq u \leq 1 \\ 1 - (2-u)^2/2; & 1 \leq u \leq 2 \end{cases}$$

Hence, $\frac{d}{du} P\{v_1 + v_2 \leq u\} = s(u)$, and the rule follows from C6.

C56. $p(\theta) = S^{-1} \cosh \theta; (0, t), S = \sinh t.$

R1. Define $A_1 = (e^t - 1)/2S$, $A_2 = (1 - e^{-t})/2S$. Set $K = \min \{k; \sum_1^k A_j \geq r_0\}$

($K = 1$ or 2). For $K = 1$, set $\theta = \ln \{1 + r_1(e^t - 1)\}$. For $K = 2$,

set $\theta = -\ln \{1 - r(1 - e^{-t})\}$ (C3, C1).

R2. Set $\theta = \ln \{Sr_0 + \sqrt{(Sr_0)^2 + 1}\}$ (C1)

C57. $p(\theta) = C^{-1} \sinh \theta$; $(0, t)$, $C = \cosh t - 1$

R. Set $\theta = \ln \{(Cr_0 + 1) + \sqrt{(Cr_0 + 1)^2 - 1}\}$

J. The rule follows from C1. Note that the (+) sign is required,

since $x \equiv e^\theta \geq 1$ and $a = (x + x^{-1})/2 \leq (x + x)/2 = x$.

C58. $f(x) = Be^{Bx}/(1 + e^{Bx})^2$; $(-\infty, \infty)$, $B > 0$.

R. Set $x = B^{-1} \ln(r_0^{-1} - 1)$ (C1)

C59. $p(y) = \frac{a_1 a_2}{a_2 - a_1} \left(y^{a_1 - 1} - y^{a_2 - 1} \right)$; $(0, 1)$, $0 < a_1 < a_2$

R. Set $y = \exp \{a_1^{-1} \ln r_1 + a_2^{-1} \ln r_2\}$

J. For $y = e^{-u}$, one finds $p(y)dy = q(u)du$ as in C18.

C60. $p(y) = \sum_1^n \frac{(a_1 \dots a_n) y^{a_i - 1}}{(a_1 - a_i) \dots (a_{i-1} - a_i) \cdot (a_{i+1} - a_i) \dots (a_n - a_i)}$;

$(0, 1)$, a_i distinct, $n \geq 2$.

R. Set $y = \exp \sum_1^n a_i^{-1} \ln r_i$

J. For $y = e^{-u}$, one has $p(y)dy = q(u)du$ as in C19.

C61. $P(\alpha'/\alpha)$

R. A direct method is given in [5,6] for sampling the Klein-Nishina differential cross section, based on C1, and an accurate fit for the inverse of the associated probability distribution.

R-INDEX

Rejection Techniques

- | | |
|---------------------------------------------------------------------------------|---------------------------------------------------------------------|
| R1. $\cos \theta, \sin \theta; p(\theta) = 2/\pi, (0, \pi/2)$ | Uniform direction, quadrant I. |
| R2. $\cos \theta, \sin \theta, \tan \theta; p(\theta) = 1/\pi, (-\pi/2, \pi/2)$ | Direction in quadrants I, IV. |
| R3. $\cos \theta, \sin \theta; p(\theta) = 1/2\pi, (0, 2\pi)$ | Uniform direction in plane, point on unit circle. |
| R4. $\Omega = (\omega_1, \omega_2, \omega_3)$ | Uniform direction in 3-space, point on unit sphere. |
| R5. $\Omega = (\omega_1, \dots, \omega_N)$ | Uniform direction in N-space, point on unit sphere $ \Omega = 1$. |
| R6. $p_1(x) \{P_2(h(x)) - P_2(g(x))\}$ | Density x distribution |
| R7. $p_1(x) h(x)$ | Density x bounded function, various versions R7A-D. |
| R8. $v^{n-1}/(\Lambda^{-1}e^v+1); n = 3/2, 1, 5/2, 2, \dots$ | (NR) non-degenerate electron gas energy. |
| R9. $x^{n-1} e^{-b\sqrt{x^2+1}}; n = 2, 3, \dots$ | (R) extreme non-degenerate electron gas momentum, Maxwell-Juttner. |
| R10. $x^{n-1}/(\Lambda^{-1}e^{a\sqrt{x^2+1}} + 1); n = 2, 3, \dots$ | (R) non-degenerate electron gas momentum. |

R11. $e^{-aE} \sinh \sqrt{bE}$

Fission energy spectrum, Cranberg density.

R12. $e^{-x^2/2}$

Normal.

R13. $P(\alpha'/\alpha, \theta)$

Polarized Klein-Nishina.

Rejection Techniques

Note. In all cases, the process is iterated until the stated condition is satisfied.

R1. $\cos \theta, \sin \theta$ for $p(\theta) = 2/\pi ; (0, \pi/2)$

R. If $S \equiv r_1^2 + r_2^2 \leq 1$, set $\cos \theta = r_1/S^{1/2}$, $\sin \theta = r_2/S^{1/2}$.

R2. $\cos \theta, \sin \theta, \tan \theta$ for $p(\theta) = 1/\pi ; (-\pi/2, \pi/2)$

R. If $S \equiv x^2 + y^2 \leq 1$, where $x = r_1, y = 2r_2 - 1$, set

$\cos \theta = x/S^{1/2}$, $\sin \theta = y/S^{1/2}$, and $\tan \theta = y/x$ ($x \neq 0$).

R3. $\cos \phi, \sin \phi$ for $p(\phi) = 1/2\pi ; (0, 2\pi)$

R1. If $S \equiv x^2 + y^2 \leq 1$, where $x = 2r_1 - 1, y = r_2$, set

$\cos \phi = (x^2 - y^2)/S$, $\sin \phi = 2xy/S$ (von Neumann)

R2. Use R1 to obtain $\cos \phi, \sin \phi$; change sign of each independently with probability 1/2.

R3. If $S \equiv x^2 + y^2 \leq 1$, where $x = 2r_1 - 1, y = 2r_2 - 1$,

set $\cos \phi = x/S^{1/2}$, $\sin \phi = y/S^{1/2}$.

R4. Uniform direction $\Omega = (\omega_1, \omega_2, \omega_3)$ in 3-space

R1. Obtain $\cos \phi$, $\sin \phi$ from R3. Set $\cos \theta = 2r_3 - 1$,

$$\sin \theta = +(1 - \cos^2 \theta)^{1/2}, \quad \Omega = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) .$$

J1. For spherical coordinates, $p(\theta, \phi) d\theta d\phi =$

$$\frac{1}{2} \sin \theta d\theta \cdot \frac{1}{2\pi} d\phi ; \quad p_1(\theta) d\theta = \frac{1}{2} \sin \theta d\theta = -\frac{1}{2} d(\cos \theta) = -\frac{1}{2} d\mu, \quad -1 \leq \mu \leq 1.$$

R2. Obtain S , $\cos \phi$, $\sin \phi$ from R3. Set $\cos \theta$

$$= 2S - 1, \quad \sin \theta = +(1 - \cos^2 \theta)^{1/2}, \quad \text{and } \Omega \text{ as in R1.}$$

J2. The density actually sampled in R3 is $p(\rho, \theta) d\rho d\theta = \rho d\rho d\theta / \pi$, with marginal density $p_1(\rho) d\rho = 2\rho d\rho$. Under the latter, the density of $S = \rho^2$ is $q(S) = 1$ on $(0, 1)$. Hence, S may be used in place of r_3 in R1.

Note. If R3 of R3 is used, one avoids unnecessary square roots by setting

$$\omega_1 = x \{S^{-1}(1 - \cos^2 \theta)\}^{1/2}, \quad \omega_2 = y \{S^{-1}(1 - \cos^2 \theta)\}^{1/2} .$$

R5. Uniform direction $\Omega = (\omega_1, \dots, \omega_N)$ in N-space.

R. If $S \equiv \sum_1^N v_i^2 \leq 1$, where $v_i = 2r_i - 1$, set $\omega_i = v_i / S^{1/2}$.

Note. Efficiency = $\pi^{N/2} / 2^{N-1} \Gamma(N/2) \rightarrow 0$, less than 1/2 for $N > 3$.
(F8). See C37 for alternative.

R6. $p(x) = A^{-1} p_1(x) \{P_2(h(x)) - P_2(g(x))\}$; (a, b) , $p_1(x)$ density on (a, b) ,

$$p_2(x) \text{ density on } (c, d), \quad P_2(y) = \int_c^y p_2(y) dy, \quad c \leq g(x) \leq h(x) \leq d.$$

R. Sample $p_1(x)$ for x on (a,b) , and $p_2(y)$ for y on (c,d) .

Accept x if $g(x) \leq y \leq h(x)$.

J. Since
$$\int_{g(x) \leq y \leq h(x)} p_1(x) dx p_2(y) dy = \int_a^b p_1(x) dx \int_{g(x)}^{h(x)} p_2(y) dy$$

$$= \int_a^b p_1(x) \{P_2(h(x)) - P_2(g(x))\} dx = A$$
, A is the probability of acceptance

(efficiency), and hence $p(x)dx$ is the probability of an accepted x lying on $(x, x+dx)$.

R7. $p(x) = A^{-1} p_1(x) h(x)$; (a,b) , $p_1(x)$ density on (a,b) , $0 \leq h(x) \leq 1$.

R. Sample $p_1(x)$ for x on (a,b) . Accept x if $r_1 \leq h(x)$.

J. Special case of R6, with $c = 0$, $d = 1$, $p_2(y) = 1$, $g(x) = 0$.

Efficiency = A .

Note. The method is useful in Klein-Nishina and Thomson (incoherent and coherent) scattering modified by form factors. Due to the nature of the latter, efficiency considerations make it advisable to take the Klein-Nishina density for $p_1(x)$ and the form factor for $h(x)$ in incoherent scattering, whereas in coherent scattering, $p_1(x)$ is based on the form factor and $h(x)$ on the Thomson cross section. For details, see a forthcoming LA report on the MCP code, which incorporates the method referred to.

R7A. $p(x) \equiv A^{-1} \cdot (b-a)^{-1} \cdot p(x)/M$; (a,b) , $M = \max p(x)$, $A = 1/(b-a)M$.

R. Accept $x \equiv a + r_0(b-a)$ if $r_1 \leq p(x)/M$.

J. Special case of R7, with $p_1(x) = 1/(b-a)$ uniform, and $h(x) = p(x)/M \leq 1$.

R7B. $p(x) = k^{-1}f(x)$; (a,b) , $k > 0$, $k = \int_a^b f(x)dx$

R. Accept $x \equiv a + r_0(b-a)$ if $r_1 \leq f(x)/D$, where $D = \max f(x)$.

J. Special case of R7A, with $h(x) = p(x)/M = k^{-1}f(x)/k^{-1}D = f(x)/D$.

Note. If $C \equiv \min f(x)$, then efficiency =

$$A = 1/(b-a)M = k/(b-a)D = \int_a^b f(x)dx/(b-a)D \geq \int_a^b \{f(x)-C\}dx/(b-a)(D-C).$$

R7C. $p(\phi) = k^{-1}f(\phi)$, $f(\phi) = K-B \cos^2 \phi$; $(0,2\pi)$, $K > B > 0$.

R1. Accept $\phi \equiv 2\pi r_0$ if $r_1 \leq f(\phi)/K$.

J1. Special case of R7B. Efficiency $\geq 1/2$. (Note, R7B)

R2. Obtain $\cos \phi$ from R3 and accept if next $r \leq f(\phi)/K$.

Note. This density occurs in polarized Compton scattering, where ϕ itself is not required.

R7D. $p(\phi) = k^{-1}f(\phi)$, $f(\phi) = K-S^2(Q \cos 2\phi + U \sin 2\phi)$

$$= K-H \cos 2(\phi-\phi_0); (0,2\pi), H = S^2(Q^2+U^2) < K,$$

$$C \equiv \min f(\phi) = K-H, D \equiv \max f(\phi) = K+H.$$

R1. Accept $\phi = 2\pi r_0$ if $r_1 \leq f(\phi)/D$.

J1. Special case of R7B. Efficiency $\geq 1/2$ (Note, R7B)

R2. Obtain $\cos \phi$, $\sin \phi$ by R3; compute $\cos 2\phi = \cos^2 \phi - \sin^2 \phi$,

$\sin 2\phi = 2 \sin \phi \cos \phi$. Accept if next $r \leq f(\phi)/D$.

Note. The density occurs in polarized Compton scattering, where ϕ itself is not required.

$$\text{R8. } \underline{p(v) = c_1^{-1} v^{n-1}/(\Lambda^{-1}e^v+1) ; (0,\infty), 0 < \Lambda \leq 1, n = 3/2, 2, 5/2, 3, \dots,}$$

$$c_1 = \zeta_a(\Lambda, n) \Gamma(n), \quad \zeta_a(\Lambda, n) \equiv \sum_1^\infty (-1)^{j+1} \Lambda^j / j^n \quad (\text{F11}).$$

R. Sample $q(v)$ for v on $(0, \infty)$ by C46; accept v if $r_0 \geq \Lambda$, or if $\Lambda > r_0 \geq \Lambda e^{-v}$.

J. One can write $p(v) = c^{-1} h(v) q(v)$, where $c^{-1} = \zeta_u(\Lambda, n) / \zeta_a(\Lambda, n)$,

$$\zeta_u(\Lambda, n) \equiv \sum_1^\infty \Lambda^{2j-1} / (2j-1)^n, \quad 0 \leq 1-\Lambda < h(v) \equiv 1-\Lambda e^{-v} < 1, \text{ and}$$

$q(v) = v^{n-1} \Lambda e^{-v} / (1-\Lambda^2 e^{-2v}) \zeta_u(\Lambda, n) \Gamma(n)$ is the density of C46. The rule follows from R7.

$$\text{R9. } \underline{\bar{p}(x) = E_b^{-1} x^{n-1} e^{-b\sqrt{x^2+1}} ; (0,\infty), b > 0, n = 2, 3, 4, \dots,}$$

$$E_b = 2^{\frac{n-1}{2}} (\Gamma(n/2) / \Gamma(1/2)) K_{\frac{n+1}{2}}(b) / b^{\frac{n-1}{2}} \quad (\text{F14}).$$

R. Sample $q(v) = D_b^{-1} v^{n-1} e^{-bv}$ for v on $(1, \infty)$ by C47; accept v with probability $h(v) = (1-1/v^2)^{\frac{n}{2}-1}$. For accepted v , set $x = (v^2-1)^{1/2}$.

J. Under the transformation $x^2+1 = v^2$, one has

$$\bar{p}(x) dx = p(v) dv \equiv (E_b^{-1} D_b) (1-1/v^2)^{\frac{n}{2}-1} (D_b^{-1} v^{n-1} e^{-bv} dv), \text{ and the rule follows}$$

from R7.

Note. For $n = 2$, $h(v) \equiv 1$, acceptance is certain, and $D_b = E_b$. For $n = 3$, $\bar{p}(x)$ is the Maxwell-Juttner relativistic momentum density [3].

R10. $\bar{p}(x) = A^{-1} x^{n-1} / (\Lambda^{-1} e^{a\sqrt{x^2+1}} + 1); (0, \infty), a > 0, 0 < \Lambda \leq 1, n = 2, 3, 4, \dots$

$$A = 2^{\frac{n-1}{2}} (\Gamma(n/2)/\Gamma(1/2)) \sum_1^{\infty} (-1)^{j+1} \Lambda^j K_{\frac{n+1}{2}}(ja) / (ja)^{\frac{n-1}{2}} \quad (F15).$$

R. Sample $\bar{q}(v)$ for v on $(1, \infty)$ by C48. Accept v if

$$r_0 \leq (1-1/v^2)^{\frac{n}{2}-1} (1-\Lambda e^{-av}). \text{ For accepted } v, \text{ set } x = (v^2-1)^{1/2}.$$

J. Under the transformation $x^2+1 = v^2$, one has $\bar{p}(x)dx = p(v)dv$

= $DA^{-1} \cdot (1-1/v^2)^{\frac{n}{2}-1} (1-\Lambda e^{-av}) \bar{q}(v)$ for the $\bar{q}(v)$ in C48, and the rule follows from R7.

R11. $\bar{p}(E) = C^{-1} e^{-aE} \sinh \sqrt{bE}; (0, \infty), a, b > 0.$

R. Define $K = 1+(b/8a)$, $L = a^{-1}\{K+\sqrt{K^2-1}\}$, $M = aL-1 > 0$.

Set $x = -\ln r_1$, $y = -\ln r_2$. Accept x if $\{y-M(x+1)\}^2 \leq bLx$. For accepted x , set $E = Lx$.

J. The rule follows from R6 and C17. Let $E = Lx$ for arbitrary $L > 0$.

$$\begin{aligned} \text{Then } \bar{p}(E)dE &= p(x)dx \equiv \frac{1}{2} C^{-1} L e^{aL-1} \cdot e^{-x} \cdot e^{-(aL-1)(x+1)} \{e^{\sqrt{bLx}} - e^{-\sqrt{bLx}}\} dx \\ &= \frac{1}{2} C^{-1} L e^M \cdot e^{-x} \cdot \{e^{-g(x)} - e^{-h(x)}\} = A^{-1} p_1(x) \{P_2(h(x)) - P_2(g(x))\} dx, \end{aligned}$$

where $M = aL-1$, $g(x) = M(x+1) - \sqrt{bLx}$, $h(x) = M(x+1) + \sqrt{bLx}$,

$p_1(x) = e^{-x}$, $p_2(y) = e^{-y}$, $P_2(y) = \int_0^y p_2(y) dy$. The above choice of L insures that $g(x) \geq 0$ on $(0, \infty)$. For a check, let $a = 3$, $b = 6$. (After Kalos [10])

R12. $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}; (0, \infty)$

R. Set $x = -\ln r_1$, $y = -\ln r_2$. Accept x if $(x-1)^2 \leq 2y$.

J. Special case of R6, with $a = 0 = c$, $b = \infty = d$, $p_1(x) = e^{-x}$,
 $p_2(y) = e^{-y}$, $P_2(y) = 1 - e^{-y}$, $g(x) = (x-1)^2/2$, $h(x) \equiv \infty$. Efficiency
 $A = (\pi/2e)^{1/2} = .76$.

Note. For $\bar{p}(x) = (2\pi)^{-1/2} e^{-x^2/2}$ on $(-\infty, \infty)$ choose (\pm) sign for accepted x above with probability 1/2.

R13. $p(\frac{\alpha'}{\alpha}, \theta)$; Polarized Compton scattering.

R. A method of sampling the Klein-Nishina cross section for polarized photons is given in [4]. This involves C61 and R7C,D. The fit used for C61 (cf. [5, 6]) is an improvement on that given in [4].

Note on Statistics

In judging the reliability of a sampling procedure, the following test may be useful. For a discrete density, pre-compute $p_i = p(i)$ for any desired set of argument values $v = i$. For a continuous density, compute $\int_{a_i}^{b_i} p(v)dv = p_i$ for a suitable set of intervals (a_i, b_i) . Let the density $p(v)$ be sampled N times, according to the rule given, and tally the number f_i of times the sample $v = i$, or $v \in (a_i, b_i)$. Fixing attention on any one index i , this yields a Bernoulli sequence of N trials, with f_i the number of "successes", and p_i the probability of success, $q_i = 1 - p_i$ the probability of "failure", in any one sampling.

In this situation, the law of large numbers states that

$P_N \equiv P\{|f_i/N - p_i| < \rho p_i\} \geq 1 - (q_i/\rho^2 p_i N) \rightarrow 1$ while the central limit theorem asserts that $P_N = \Phi(\rho\sqrt{p_i N/q_i}) + Z_N$, where $\Phi(z) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz$, and $Z_N \rightarrow 0$.

One may note that, insofar as a method is correct, its statistical reliability depends only on the density itself.

REFERENCES

1. K. M. Case, F. de Hoffmann, G. Placzek, Introduction to the Theory of Neutron Diffusion, Vol. I (Los Alamos Scientific Laboratory, 1953) p. 153.
2. E. D. Cashwell, C. J. Everett, G. D. Turner, "A Method of Sampling Certain Probability Densities Without Inversion of Their Distribution Functions," Los Alamos Scientific Laboratory report LAMS (in preparation).
3. S. Chandrasekhar, An Introduction to the Study of Stellar Structure (Dover Publications, Inc., 1958) Ch. X.
4. C. J. Everett, "A Relativity Notebook for Monte Carlo Practice," Los Alamos Scientific Laboratory report LA-3839 (1968).
5. C. J. Everett, E. D. Cashwell, "Approximation for the Inverse of the Klein-Nishina Probability Distribution," Los Alamos Scientific Laboratory report LA-4448 (1970).
6. C. J. Everett, E. D. Cashwell, G. D. Turner, "A New Method of Sampling the Klein-Nishina Probability Distribution for All Incident Photon Energies Above 1 keV," Los Alamos Scientific Laboratory report LA-4663 (1971).
7. C. J. Everett, P. R. Stein, "On Random Sequences of Integers," Bull. Amer. Math. Soc., 76, 349-351 (1970).
8. C. J. Everett, P. R. Stein, "On Random Sequences of Integers," Los Alamos Scientific Laboratory report LA-4268 (1969).
9. H. Kahn, "Applications of Monte Carlo," Rand Corporation report RM-1237-AEC (1956).
10. M. H. Kalos, F. R. Nakache, J. Celnik, Monte Carlo Methods in Reactor Computations, Computing Methods in Reactor Physics (Gordon and Breach, N. Y., 1968) Ch. 5.
11. M. H. Kalos, "Monte Carlo Calculations of the Ground State of Three- and Four-Body Nuclei," Phys. Rev. 128, 1791-1795 (1962).
12. G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge Univ. Press, 1952) pp. 183, 185.