Numerične aproksimacije funkcij

14. 2. 2023

Zgled: Izračunaj e^5 ? (kalkulator: 148.4135)

Iz matematike vemo, da lahko e^x zapišemo z neskončno vrsto:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Intermezzo:

Vrsta je konvergentna zej velja,

a.) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$; $\forall x$

b.) $\exists n_0$, $\frac{x^n}{n!} < \frac{1}{2^{n-1}}$; $\forall n > n_0$

ker je geometrijska vrsta konvergentna je konvergentna tudi naša vrsta.

Težava je, da na računalnikih ne moremo seštevati do ∞. Seštejemo do končnega N.

$$e^x \simeq \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + \sum_{n=1}^{N} \frac{x^n}{n!} = \exp(x, N)$$

Do katerega N seštevamo ? Seštevati moramo tako dolgo, dokler napaka $\varepsilon(x)$ manjša od želene natančnosti.

$$\left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| < \varepsilon(x) \sim 10^{-15} \longrightarrow$$

Tega ne poznamo!

Nasvet: Numerični račun poskušamo narediti na čim bolj učinkovit način:

$$\exp(x, N) = 1 + \sum_{n=1}^{N} \underbrace{\frac{x^n}{n!}}_{a_n}$$

$n = 1: \quad a_1 = \frac{x}{1}$

$n = 2: \quad a_2 = \frac{x^2}{2!} = \frac{x}{2} \cdot a_1$

$n = 3: \quad a_3 = \frac{x^3}{3!} = \frac{x}{3} \cdot a_2$

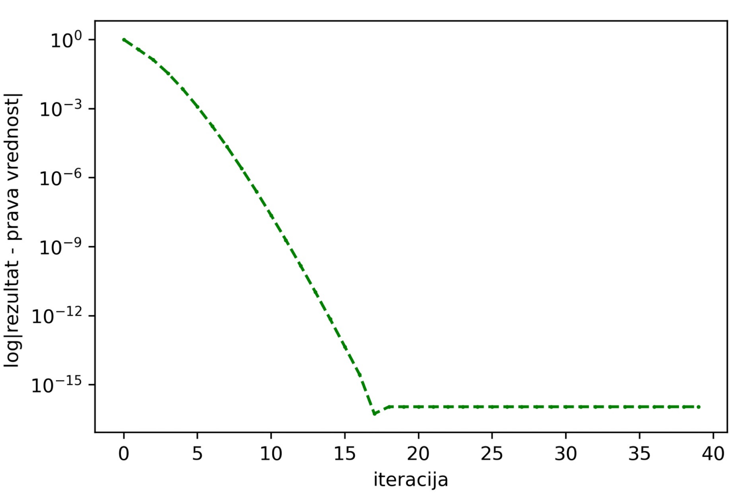
$\vdots$

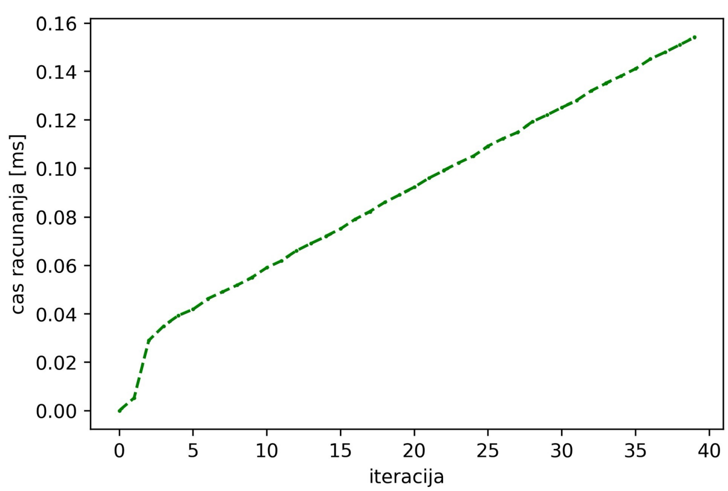
$$\boxed{a_n = a_{n-1} \cdot \frac{x}{n}}$$

S takšnim preprostim rekurzivnim klicem se ognemo računanja pow(x,n) in programiranja n!

Seštevamo tako dolgo, da je $a_n < \varepsilon$.

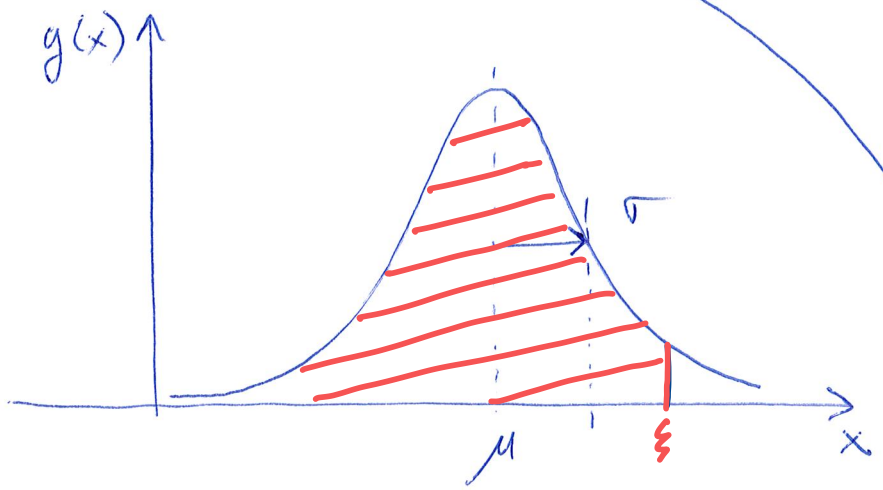
Zgled: "ExpCasovnaZahtevnost.py"





Gaussova verjetnostna porazdelitev:

$$g(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Normalizacija.

Mera

Velja:

$$\int_{-\infty}^{\infty} g(x)\,dx = 1$$

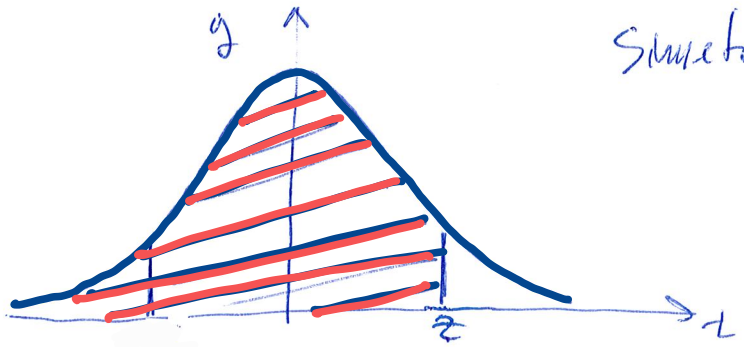
(Funkcija je normirana!)

Vpeljemo nova spremenljivko: $t = \frac{x-\mu}{\sqrt{2}\sigma}$

$$dt = \frac{1}{\sqrt{2}\sigma}\,dx$$

$$G(z) = \int_{-\infty}^{z} g(t)\,dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-t^2}\,dt$$

$$z = \frac{\xi-\mu}{\sqrt{2}\sigma}$$



Simetrično na y-os

$$G_1(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{\pi}} e^{-t^2} dt = \left| \text{Rozpisemy na wcześniej wzory} \right|$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-t^2} dt\right] = \left| \text{Wiedzemy} - Erf(z) \right|$$

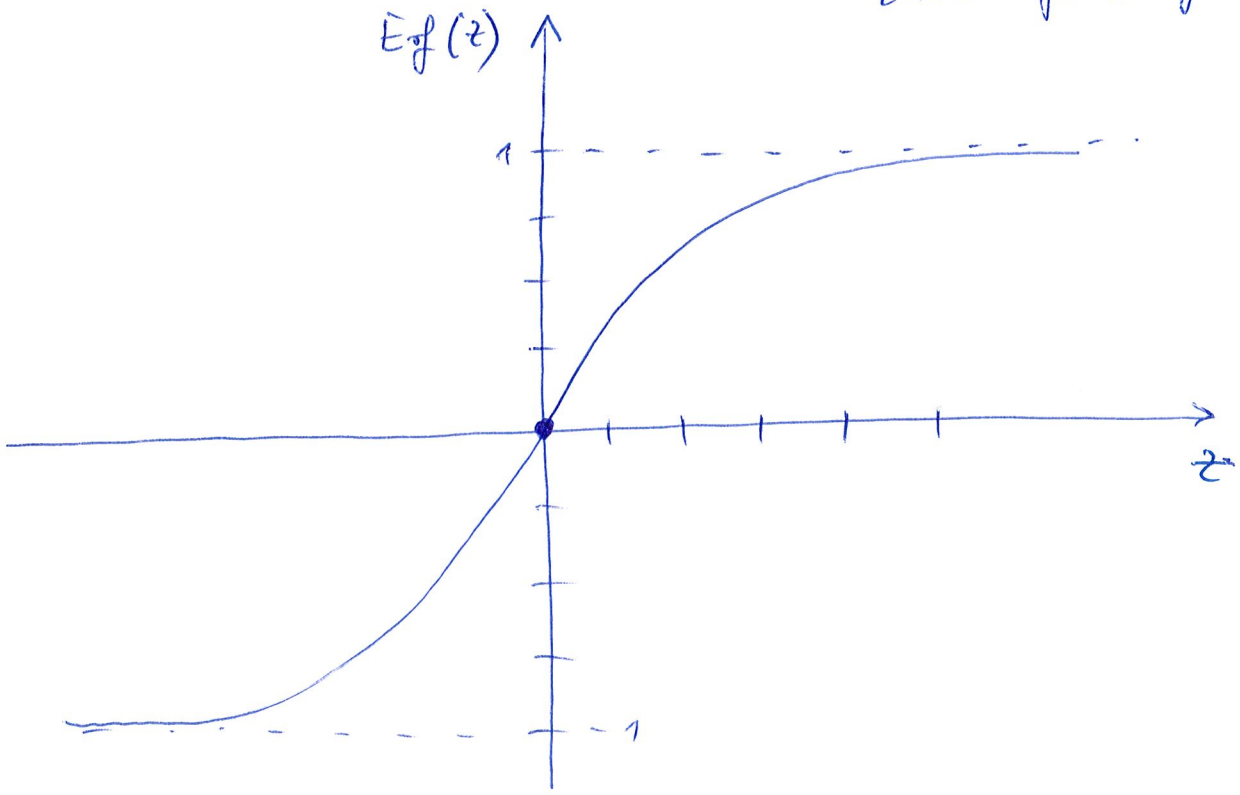
$$= \frac{1}{2}\left[Erf(z) - Erf(-\infty)\right] =$$

$$= \frac{1}{2}\left[1 + Erf(z)\right] = \frac{1}{2}\left[1 + Erf\left(\frac{\xi - \mu}{\sqrt{2}\sigma}\right)\right]$$

(*)

Uvedemo: $\frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = Erf(z)$

"Error function" ali "Errorjeva funkcija"



Ta integral bodemo sedaj izračunali:

$$Erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

S čim si bi lahko pomagali?

Vemo: $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \dots$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(t)^{2n} (-1)^n}{n!}$$

Potem:

$$\underline{Erf(z)} = \frac{2}{\sqrt{\pi}} \int_0^z \sum_{n=0}^{\infty} \frac{t^{2n}(-1)^n}{n!}\, dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^z t^{2n}\, dt =$$

Pazi, kako seštevamo vrsto!

$$= \underline{\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{z^{2n+1}}{2n+1}} \quad (*)$$

Računamo v neskončno vrsto.

Z računalnikom ne bomo mogli seštevati do ∞.
Koliko členov potrebujemo, da dosežemo želeno natančnost?

$$\left| \frac{(-1)^n}{n!} \frac{z^{2n+1}}{2n+1} \frac{2}{\sqrt{\pi}} \right| < \varepsilon \left(= \underbrace{10^{-6}}_{\text{Mpr.}}\right)$$

Kje dobro deluje ta približek?

Asimptotska vrsta:

$$Erfc(z) = 1 - Erf(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}\, dt$$

Vpeljemo: $t^2 = u$, $du = 2t\, dt$ oziroma $dt = \frac{du}{2\sqrt{u}}$

$$= \frac{2}{\sqrt{\pi}} \int_{z^2}^{\infty} \underbrace{e^{-u}}_{dv} \underbrace{\frac{1}{2\sqrt{u}}}_{w} du = \frac{1}{\sqrt{\pi}} \left[-\frac{e^{-u}}{\sqrt{u}} \Big|_{z^2}^{\infty} - \frac{1}{2} \int_{z^2}^{\infty} \underbrace{e^{-u}}_{dv} \underbrace{\frac{1}{u^{3/2}}}_{w} du \right]$$

↑ Per Partes

$$= \frac{1}{\sqrt{\pi}} \left[\frac{e^{-z^2}}{z} + \frac{1}{2} \frac{e^{-u}}{u^{3/2}} \Big|_{z^2}^{\infty} + \frac{1}{2}\frac{3}{2} \int_{z^2}^{\infty} e^{-u} \frac{1}{u^{5/2}} du \right] =$$

$$= \frac{1}{\sqrt{\pi}} \left[\frac{e^{-z^2}}{z} - \frac{1}{2} \frac{e^{-z^2}}{z^3} - \frac{3}{2^2} \frac{e^{-u}}{u^{5/2}} \Big|_{z^2}^{\infty} - \frac{3}{2^2} \int_{z^2}^{\infty} \frac{5}{2} e^{-u} \frac{1}{u^{7/2}} du \right] =$$

$$= \frac{1}{\sqrt{\pi}} \left[\frac{e^{-z^2}}{z} - \frac{1}{2} \frac{e^{-z^2}}{z^3} + \frac{3}{2^2} \frac{e^{-z^2}}{z^5} + \frac{3 \cdot 5}{2^3} \frac{e^{-u}}{u^{7/2}} \Big|_{z^2}^{\infty} + \int_{z^2}^{\infty} \frac{3 \cdot 5 \cdot 7}{2^4} \frac{e^{-u}}{u^{9/2}} du \right] =$$

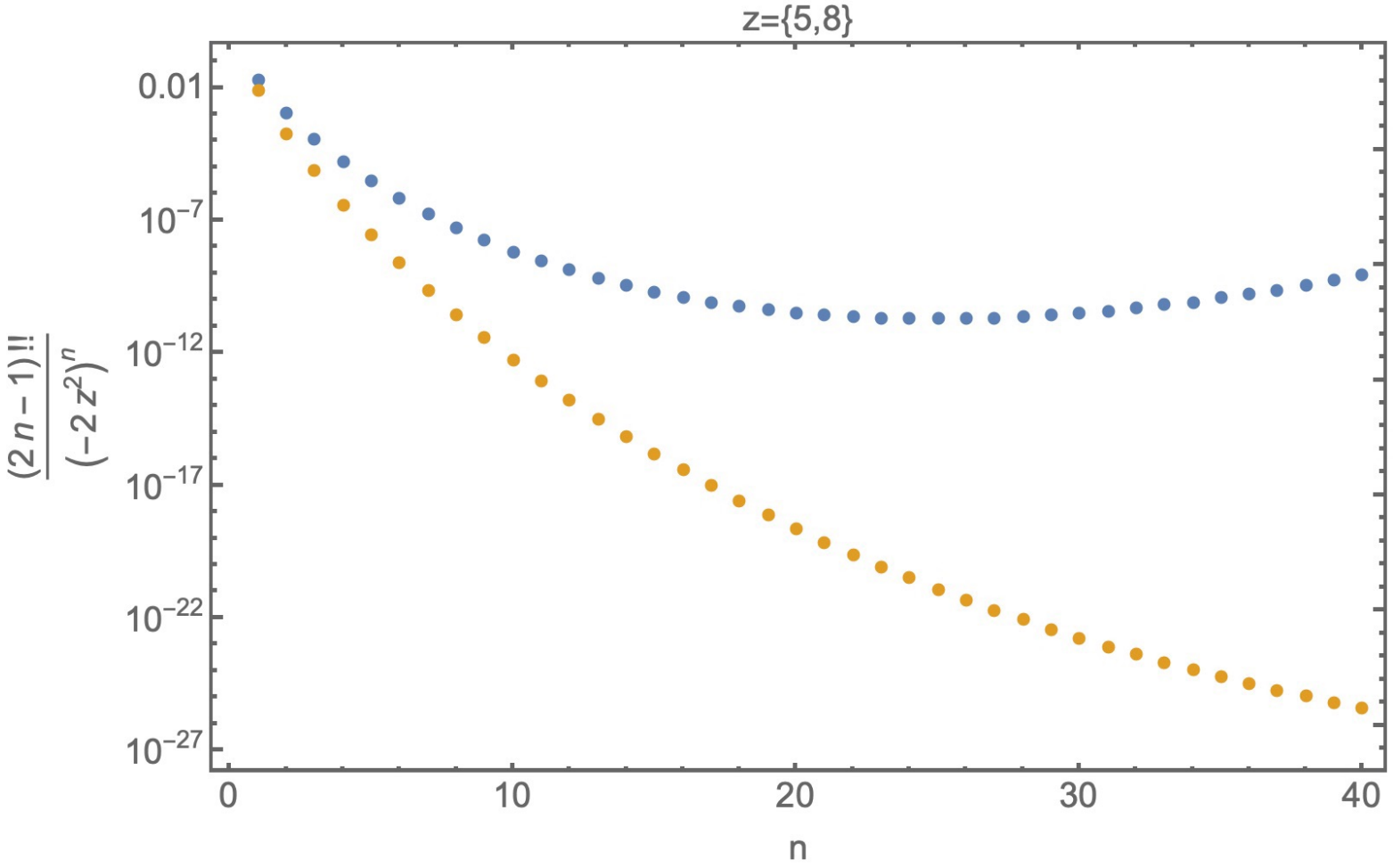
$$= \frac{1}{\sqrt{\pi}} \left[\frac{e^{-z^2}}{z} - \frac{1}{2} \frac{e^{-z^2}}{z^3} + \frac{1 \cdot 3}{2^2} \frac{e^{-z^2}}{z^5} - \frac{1 \cdot 3 \cdot 5}{2^3} \frac{e^{-z^2}}{z^7} + \dots \right]$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{e^{-z^2}}{z} \left[1 - \frac{1}{2z^2} + \frac{1 \cdot 3}{2^2 \cdot z^4} - \frac{1 \cdot 3 \cdot 5}{2^3 z^7} + \dots \right] =$$

$$= \frac{1}{\sqrt{\pi}} \frac{e^{-z^2}}{z} \left[1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(-2z^2)^n} \right] \quad (**)$$

Te vsote ne bomo seštevali do ∞. Dokazno problematiko bomo obravnavali le kvalitativno. Problematika bo dobro definirana za $z \gg 0$!

Imamo dva dobra problema za majhne in velike z. Kaj pa vmes?


z={5,8}
0.01
10^-7
10^-12
10^-17
10^-22
10^-27
0
10
20
30
40
n
(2n-1)!!/(-2z^2)^n

Uporabimo Poenostavljeno aproksimacijo: A.S.

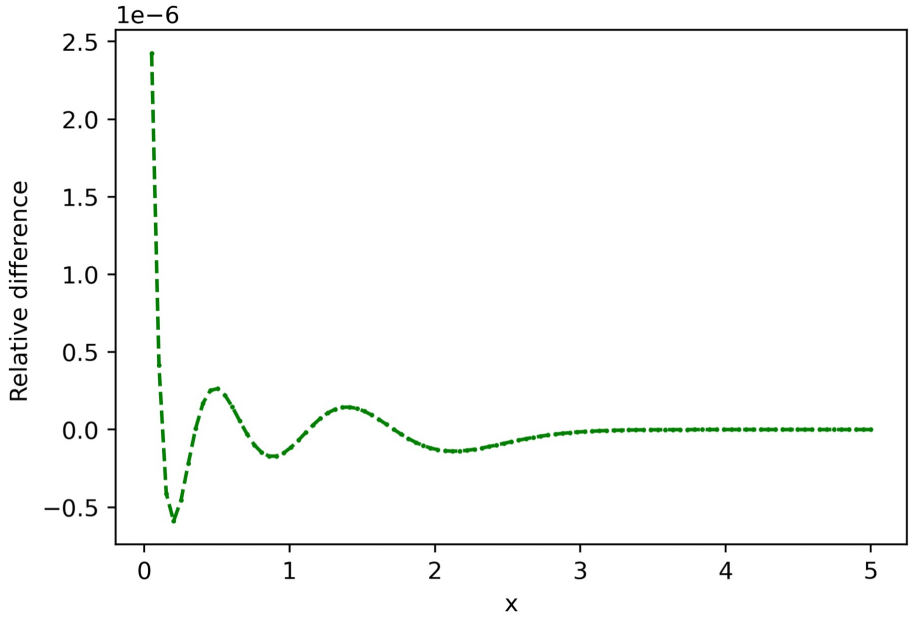
$$Erf(x) = 1 - e^{-x^2}\left(a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5\right) + \varepsilon(x)$$

$$t = \frac{1}{1 + p x}$$

$$p = 0.3275911$$
$$a_1 = 0.254829592$$
$$a_2 = -0.284496736$$
$$a_3 = 1.421413741$$
$$a_4 = -1.453152027$$
$$a_5 = 1.061405429$$

Za približek velja $|\varepsilon(x)| \leq 1.5 \cdot 10^{-7}$ za $\forall x \in [0, \infty)$.

Zgled: ErrorFunction.py



7. Error Function and Fresnel Integrals

Mathematical Properties

7.1. Error Function

Definitions

7.1.1 $$\operatorname{erf} z=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

7.1.2 $$\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt=1-\operatorname{erf} z$$

7.1.3 $$w(z)=e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right)=e^{-z^2}\operatorname{erfc}(-iz)$$

In **7.1.2** the path of integration is subject to the restriction $\arg t\to\alpha$ with $|\alpha|<\frac{\pi}{4}$ as $t\to\infty$ along the path. ($\alpha=\frac{\pi}{4}$ is permissible if $\mathscr{R}t^2$ remains bounded to the left.)

Integral Representation

7.1.4

$$w(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}dt}{z-t}=\frac{2iz}{\pi}\int_0^\infty\frac{e^{-t^2}dt}{z^2-t^2}\qquad(\mathscr{I}z>0)$$

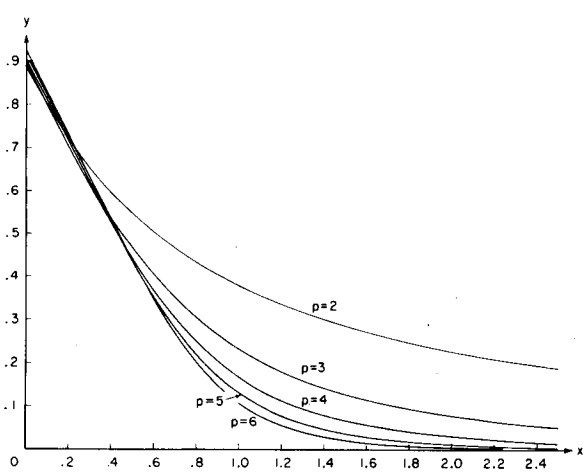


FIGURE 7.1. $y=e^{x^p}\int_x^\infty e^{-t^p}dt$.

$p=2(1)6$

Series Expansions

7.1.5 $$\operatorname{erf} z=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

7.1.6 $$=\frac{2}{\sqrt{\pi}}e^{-z^2}\sum_{n=0}^{\infty}\frac{2^n}{1\cdot 3\cdots(2n+1)}z^{2n+1}$$

7.1.7 $$=\sqrt{2}\sum_{n=0}^{\infty}(-1)^n[I_{2n+1/2}(z^2)-I_{2n+3/2}(z^2)]$$

7.1.8 $$w(z)=\sum_{n=0}^{\infty}\frac{(iz)^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

For $I_{n-\frac{1}{2}}(x)$, see chapter **10.**

Symmetry Relations

7.1.9 $$\operatorname{erf}(-z)=-\operatorname{erf} z$$

7.1.10 $$\operatorname{erf}\bar{z}=\overline{\operatorname{erf} z}$$

7.1.11 $$w(-z)=2e^{-z^2}-w(z)$$

7.1.12 $$w(\bar{z})=\overline{w(-z)}$$

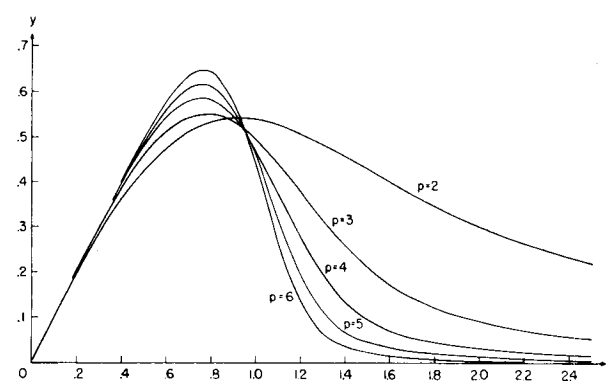


FIGURE 7.2. $y=e^{-x^p}\int_0^x e^{t^p}dt$.

$p=2(1)6$

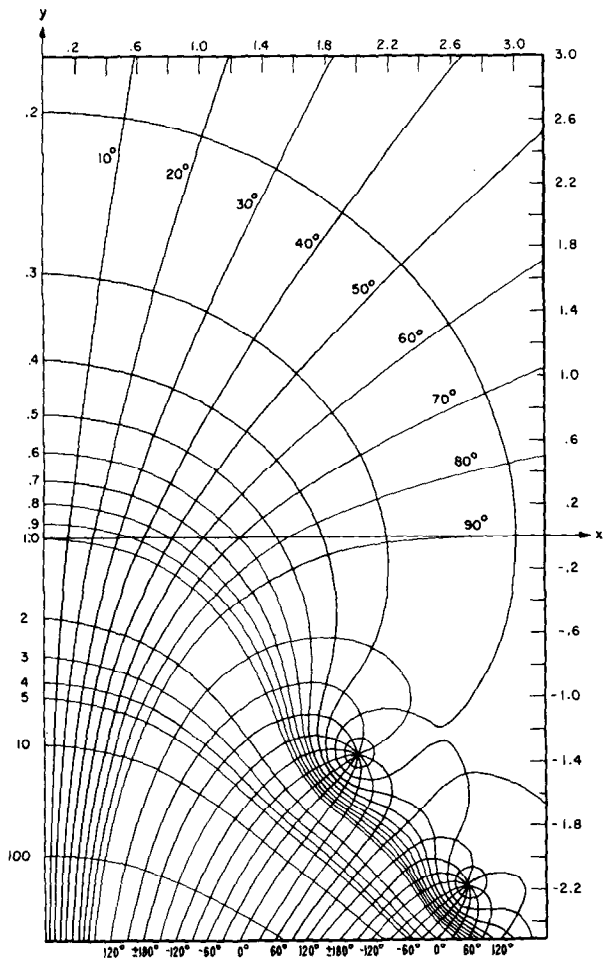


FIGURE 7.3. *Altitude Chart of* $w(z)$.

Inequalities [7.11], [7.17]

7.1.13

$$\frac{1}{x+\sqrt{x^2+2}}<e^{x^2}\int_x^\infty e^{-t^2}dt\leq\frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}}\qquad(x\geq 0)$$

(For other inequalities see [7.2].)

Continued Fractions

7.1.14

$$2e^{z^2}\int_z^\infty e^{-t^2}dt=\frac{1}{z+}\,\frac{1/2}{z+}\,\frac{1}{z+}\,\frac{3/2}{z+}\,\frac{2}{z+}\cdots\qquad(\Re z>0)$$

7.1.15

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2}dt}{z-t}=\frac{1}{z-}\,\frac{1/2}{z-}\,\frac{1}{z-}\,\frac{3/2}{z-}\,\frac{2}{z-}\cdots$$
$$=\frac{1}{\sqrt{\pi}}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{H_k^{(n)}}{z-x_k^{(n)}}\qquad(\Im z\neq 0)$$

$x_k^{(n)}$ and $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials. For numerical values see chapter **25**.

Value at Infinity

7.1.16 $\operatorname{erf} z\to 1\left(z\to\infty \text{ in } |\arg z|<\frac{\pi}{4}\right)$

Maximum and Inflection Points for Dawson's Integral [7.31]

$$F(x)=e^{-x^2}\int_0^x e^{t^2}dt$$

7.1.17 $F(.92413\ 88730\ldots)=.54104\ 42246\ldots$

7.1.18 $F(1.50197\ 52682\ldots)=.42768\ 66160\ldots$

Derivatives

7.1.19

$$\frac{d^{n+1}}{dz^{n+1}}\operatorname{erf} z=(-1)^n\frac{2}{\sqrt{\pi}}H_n(z)e^{-z^2}\qquad(n=0,1,2,\ldots)$$

7.1.20

$$w^{(n+2)}(z)+2zw^{(n+1)}(z)+2(n+1)w^{(n)}(z)=0\qquad(n=0,1,2,\ldots)$$

$$w^{(0)}(z)=w(z),\qquad w'(z)=-2zw(z)+\frac{2i}{\sqrt{\pi}}$$

(For the Hermite polynomials $H_n(z)$ see chapter **22**.)

Relation to Confluent Hypergeometric Function (see chapter **13**)

7.1.21

$$\operatorname{erf} z=\frac{2z}{\sqrt{\pi}}M\left(\frac{1}{2},\frac{3}{2},-z^2\right)=\frac{2z}{\sqrt{\pi}}e^{-z^2}M\left(1,\frac{3}{2},z^2\right)$$

The Normal Distribution Function With Mean m and Standard Deviation σ (see chapter **26**)

7.1.22

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{(t-m)^2}{2\sigma^2}}dt=\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x-m}{\sigma\sqrt{2}}\right)\right)$$

Asymptotic Expansion

7.1.23

$$\sqrt{\pi}ze^{z^2}\operatorname{erfc} z\sim 1+\sum_{m=1}^{\infty}(-1)^m\frac{1\cdot 3\ldots(2m-1)}{(2z^2)^m}$$
$$\left(z\to\infty,\ |\arg z|<\frac{3\pi}{4}\right)$$

If $R_n(z)$ is the remainder after n terms then

7.1.24

$$R_n(z)=(-1)^n\frac{1\cdot 3\ldots(2n-1)}{(2z^2)^n}\theta,$$

$$\theta=\int_0^\infty e^{-t}\left(1+\frac{t}{z^2}\right)^{-n-\frac{1}{2}}dt \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

$$|\theta|<1 \qquad \left(|\arg z|<\frac{\pi}{4}\right)$$

For x real, $R_n(x)$ is less in absolute value than the first neglected term and of the same sign.

Rational Approximations [2] $(0\le x<\infty)$

7.1.25

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3)\,e^{-x^2}+\epsilon(x), \qquad t=\frac{1}{1+px}$$

$$|\epsilon(x)|\le 2.5\times 10^{-5}$$

$p=.47047$ $\quad a_1=.34802\ 42$ $\quad a_2=-.09587\ 98$ $\quad a_3=.74785\ 56$

7.1.26

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5)e^{-x^2}+\epsilon(x),$$

$$t=\frac{1}{1+px}$$

$$|\epsilon(x)|\le 1.5\times 10^{-7}$$

$p=.32759\ 11$ $\quad a_1=.25482\ 9592$

$a_2=-.28449\ 6736$ $\quad a_3=1.42141\ 3741$

$a_4=-1.45315\ 2027$ $\quad a_5=1.06140\ 5429$

7.1.27

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+a_3x^3+a_4x^4]^4}+\epsilon(x)$$

$$|\epsilon(x)|\le 5\times 10^{-4}$$

$a_1=.278393$ $\quad a_2=.230389$

$a_3=.000972$ $\quad a_4=.078108$

7.1.28

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+\cdots+a_6x^6]^{16}}+\epsilon(x)$$

$$|\epsilon(x)|\le 3\times 10^{-7}$$

$a_1=.07052\ 30784$ $\quad a_2=.04228\ 20123$

$a_3=.00927\ 05272$ $\quad a_4=.00015\ 20143$

$a_5=.00027\ 65672$ $\quad a_6=.00004\ 30638$

[2] Approximations **7.1.25–7.1.28** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N. J., 1955 (with permission).

Infinite Series Approximation for Complex Error Function [7.19]

7.1.29

$$\operatorname{erf}(x+iy)=\operatorname{erf} x+\frac{e^{-x^2}}{2\pi x}[(1-\cos 2xy)+i\sin 2xy]$$

$$+\frac{2}{\pi}e^{-x^2}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{n^2+4x^2}[f_n(x,y)+ig_n(x,y)]+\epsilon(x,y)$$

where

$$f_n(x,y)=2x-2x\cosh ny\cos 2xy+n\sinh ny\sin 2xy$$

$$g_n(x,y)=2x\cosh ny\sin 2xy+n\sinh ny\cos 2xy$$

$$|\epsilon(x,y)|\approx 10^{-16}|\operatorname{erf}(x+iy)|$$

7.2. Repeated Integrals of the Error Function

Definition

7.2.1

$$\mathrm{i}^n\operatorname{erfc} z=\int_z^\infty \mathrm{i}^{n-1}\operatorname{erfc} t\,dt \qquad (n=0,1,2,\ldots)$$

$$\mathrm{i}^{-1}\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}e^{-z^2},\quad \mathrm{i}^0\operatorname{erfc} z=\operatorname{erfc} z$$

Differential Equation

7.2.2

$$\frac{d^2y}{dz^2}+2z\frac{dy}{dz}-2ny=0$$

$$y=A\mathrm{i}^n\operatorname{erfc} z+B\mathrm{i}^n\operatorname{erfc}(-z)$$

(A and B are constants.)

Expression as a Single Integral

7.2.3

$$\mathrm{i}^n\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}\int_z^\infty\frac{(t-z)^n}{n!}e^{-t^2}dt$$

Power Series [3]

7.2.4

$$\mathrm{i}^n\operatorname{erfc} z=\sum_{k=0}^{\infty}\frac{(-1)^kz^k}{2^{n-k}k!\,\Gamma\left(1+\frac{n-k}{2}\right)}$$

Recurrence Relations

7.2.5

$$\mathrm{i}^n\operatorname{erfc} z=-\frac{z}{n}\mathrm{i}^{n-1}\operatorname{erfc} z+\frac{1}{2n}\mathrm{i}^{n-2}\operatorname{erfc} z$$

$$(n=1,2,3,\ldots)$$

7.2.6

$$2(n+1)(n+2)\mathrm{i}^{n+2}\operatorname{erfc} z$$

$$=(2n+1+2z^2)\mathrm{i}^n\operatorname{erfc} z-\frac{1}{2}\mathrm{i}^{n-2}\operatorname{erfc} z$$

$$(n=1,2,3,\ldots)$$

[3] The terms in this series corresponding to $k=n+2$, $n+4$, $n+6$, . . . are understood to be zero.

Table 7.1 ERROR FUNCTION AND ITS DERIVATIVE

x	$\frac{2}{\sqrt{\pi}} e^{-x^2}$	erf x	x	$\frac{2}{\sqrt{\pi}} e^{-x^2}$	erf x
0.00	1.12837 91671	0.00000 00000	0.50	0.87878 25789	0.52049 98778
0.01	1.12826 63348	0.01128 34156	0.51	0.86995 15467	0.52924 36198
0.02	1.12792 79057	0.02256 45747	0.52	0.86103 70343	0.53789 86305
0.03	1.12736 40827	0.03384 12223	0.53	0.85204 34444	0.54646 40969
0.04	1.12657 52040	0.04511 11061	0.54	0.84297 51813	0.55493 92505
0.05	1.12556 17424	0.05637 19778	0.55	0.83383 66473	0.56332 33663
0.06	1.12432 43052	0.06762 15944	0.56	0.82463 22395	0.57161 57638
0.07	1.12286 36333	0.07885 77198	0.57	0.81536 63461	0.57981 58062
0.08	1.12118 06004	0.09007 81258	0.58	0.80604 33431	0.58792 29004
0.09	1.11927 62126	0.10128 05939	0.59	0.79666 75911	0.59593 64972
0.10	1.11715 16068	0.11246 29160	0.60	0.78724 34317	0.60385 60908
0.11	1.11480 80500	0.12362 28962	0.61	0.77777 51846	0.61168 12189
0.12	1.11224 69379	0.13475 83518	0.62	0.76826 71442	0.61941 14619
0.13	1.10946 97934	0.14586 71148	0.63	0.75872 35764	0.62704 64433
0.14	1.10647 82654	0.15694 70331	0.64	0.74914 87161	0.63458 58291
0.15	1.10327 41267	0.16799 59714	0.65	0.73954 67634	0.64202 93274
0.16	1.09985 92726	0.17901 18132	0.66	0.72992 18814	0.64937 66880
0.17	1.09623 57192	0.18999 24612	0.67	0.72027 81930	0.65662 77023
0.18	1.09240 56008	0.20093 58390	0.68	0.71061 97784	0.66378 22027
0.19	1.08837 11683	0.21183 98922	0.69	0.70095 06721	0.67084 00622
0.20	1.08413 47871	0.22270 25892	0.70	0.69127 48604	0.67780 11938
0.21	1.07969 89342	0.23352 19230	0.71	0.68159 62792	0.68466 55502
0.22	1.07506 61963	0.24429 59116	0.72	0.67191 88112	0.69143 31231
0.23	1.07023 92672	0.25502 25996	0.73	0.66224 62838	0.69810 39429
0.24	1.06522 09449	0.26570 00590	0.74	0.65258 24665	0.70467 80779
0.25	1.06001 41294	0.27632 63902	0.75	0.64293 10692	0.71115 56337
0.26	1.05462 18194	0.28689 97232	0.76	0.63329 57399	0.71753 67528
0.27	1.04904 71098	0.29741 82185	0.77	0.62368 00626	0.72382 16140
0.28	1.04329 31885	0.30788 00680	0.78	0.61408 75556	0.73001 04313
0.29	1.03736 33334	0.31828 34959	0.79	0.60452 16696	0.73610 34538
0.30	1.03126 09096	0.32862 67595	0.80	0.59498 57863	0.74210 09647
0.31	1.02498 93657	0.33890 81503	0.81	0.58548 32161	0.74800 32806
0.32	1.01855 22310	0.34912 59948	0.82	0.57601 71973	0.75381 07509
0.33	1.01195 31119	0.35927 86550	0.83	0.56659 08944	0.75952 37569
0.34	1.00519 56887	0.36936 45293	0.84	0.55720 73967	0.76514 27115
0.35	0.99828 37121	0.37938 20536	0.85	0.54786 97173	0.77066 80576
0.36	0.99122 10001	0.38932 97011	0.86	0.53858 07918	0.77610 02683
0.37	0.98401 14337	0.39920 59840	0.87	0.52934 34773	0.78143 98455
0.38	0.97665 89542	0.40900 94534	0.88	0.52016 05514	0.78668 73192
0.39	0.96916 75592	0.41873 87001	0.89	0.51103 47116	0.79184 32468
0.40	0.96154 12988	0.42839 23550	0.90	0.50196 85742	0.79690 82124
0.41	0.95378 42727	0.43796 90902	0.91	0.49296 46742	0.80188 28258
0.42	0.94590 06256	0.44746 76184	0.92	0.48402 54639	0.80676 77215
0.43	0.93789 45443	0.45688 66945	0.93	0.47515 33132	0.81156 35586
0.44	0.92977 02537	0.46622 51153	0.94	0.46635 05090	0.81627 10190
0.45	0.92153 20130	0.47548 17198	0.95	0.45761 92546	0.82089 08073
0.46	0.91318 41122	0.48465 53900	0.96	0.44896 16700	0.82542 36496
0.47	0.90473 08685	0.49374 50509	0.97	0.44037 97913	0.82987 02930
0.48	0.89617 66223	0.50274 96707	0.98	0.43187 55710	0.83423 15043
0.49	0.88752 57337	0.51166 82612	0.99	0.42345 08779	0.83850 80696
0.50	0.87878 25789	0.52049 98778	1.00	0.41510 74974	0.84270 07929
	$\left[\begin{smallmatrix}(-5)3\\5\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-5)1\\5\end{smallmatrix}\right]$		$\left[\begin{smallmatrix}(-5)1\\5\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-5)1\\5\end{smallmatrix}\right]$

See Example 1.

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad \frac{\sqrt{\pi}}{2} = 0.88622\ 69255$$

ERROR FUNCTION AND ITS DERIVATIVE

Table 7.1

x	$\frac{2}{\sqrt{\pi}} e^{-x^2}$	erf x	x	$\frac{2}{\sqrt{\pi}} e^{-x^2}$	erf x
1.00	0.41510 74974	0.84270 07929	1.50	0.11893 02892	0.96610 51465
1.01	0.40684 71315	0.84681 04962	1.51	0.11540 38270	0.96727 67481
1.02	0.39867 13992	0.85083 80177	1.52	0.11195 95356	0.96841 34969
1.03	0.39058 18368	0.85478 42115	1.53	0.10859 63195	0.96951 62091
1.04	0.38257 98986	0.85864 99465	1.54	0.10531 30683	0.97058 56899
1.05	0.37466 69570	0.86243 61061	1.55	0.10210 86576	0.97162 27333
1.06	0.36684 43034	0.86614 35866	1.56	0.09898 19506	0.97262 81220
1.07	0.35911 31488	0.86977 32972	1.57	0.09593 17995	0.97360 26275
1.08	0.35147 46245	0.87332 61584	1.58	0.09295 70461	0.97454 70093
1.09	0.34392 97827	0.87680 31019	1.59	0.09005 65239	0.97546 20158
1.10	0.33647 95978	0.88020 50696	1.60	0.08722 90586	0.97634 83833
1.11	0.32912 49667	0.88353 30124	1.61	0.08447 34697	0.97720 68366
1.12	0.32186 67103	0.88678 78902	1.62	0.08178 85711	0.97803 80884
1.13	0.31470 55742	0.88997 06704	1.63	0.07917 31730	0.97884 28397
1.14	0.30764 22299	0.89308 23276	1.64	0.07662 60821	0.97962 17795
1.15	0.30067 72759	0.89612 38429	1.65	0.07414 61034	0.98037 55850
1.16	0.29381 12389	0.89909 62029	1.66	0.07173 20405	0.98110 49213
1.17	0.28704 45748	0.90200 03990	1.67	0.06938 26972	0.98181 04416
1.18	0.28037 76702	0.90483 74269	1.68	0.06709 68781	0.98249 27870
1.19	0.27381 08437	0.90760 82860	1.69	0.06487 33895	0.98315 25869
1.20	0.26734 43470	0.91031 39782	1.70	0.06271 10405	0.98379 04586
1.21	0.26097 83664	0.91295 55080	1.71	0.06060 86436	0.98440 70075
1.22	0.25471 30243	0.91553 38810	1.72	0.05856 50157	0.98500 28274
1.23	0.24854 83805	0.91805 01041	1.73	0.05657 89788	0.98557 84998
1.24	0.24248 44335	0.92050 51843	1.74	0.05464 93607	0.98613 45950
1.25	0.23652 11224	0.92290 01283	1.75	0.05277 49959	0.98667 16712
1.26	0.23065 83281	0.92523 59418	1.76	0.05095 47262	0.98719 02752
1.27	0.22489 58748	0.92751 36293	1.77	0.04918 74012	0.98769 09422
1.28	0.21923 35317	0.92973 41930	1.78	0.04747 18791	0.98817 41959
1.29	0.21367 10145	0.93189 86327	1.79	0.04580 70274	0.98864 05487
1.30	0.20820 79868	0.93400 79449	1.80	0.04419 17233	0.98909 05016
1.31	0.20284 40621	0.93606 31228	1.81	0.04262 48543	0.98952 45446
1.32	0.19757 88048	0.93806 51551	1.82	0.04110 53185	0.98994 31565
1.33	0.19241 17326	0.94001 50262	1.83	0.03963 20255	0.99034 68051
1.34	0.18734 23172	0.94191 37153	1.84	0.03820 38966	0.99073 59476
1.35	0.18236 99865	0.94376 21961	1.85	0.03681 98653	0.99111 10301
1.36	0.17749 41262	0.94556 14366	1.86	0.03547 88774	0.99147 24883
1.37	0.17271 40811	0.94731 23980	1.87	0.03417 98920	0.99182 07476
1.38	0.16802 91568	0.94901 60353	1.88	0.03292 18811	0.99215 62228
1.39	0.16343 86216	0.95067 32958	1.89	0.03170 38307	0.99247 93184
1.40	0.15894 17077	0.95228 51198	1.90	0.03052 47404	0.99279 04292
1.41	0.15453 76130	0.95385 24394	1.91	0.02938 36241	0.99308 99398
1.42	0.15022 55027	0.95537 61786	1.92	0.02827 95101	0.99337 82251
1.43	0.14600 45107	0.95685 72531	1.93	0.02721 14412	0.99365 56502
1.44	0.14187 37413	0.95829 65696	1.94	0.02617 84752	0.99392 25709
1.45	0.13783 22708	0.95969 50256	1.95	0.02517 96849	0.99417 93336
1.46	0.13387 91486	0.96105 35095	1.96	0.02421 41583	0.99442 62755
1.47	0.13001 33993	0.96237 28999	1.97	0.02328 09986	0.99466 37246
1.48	0.12623 40239	0.96365 40654	1.98	0.02237 93244	0.99489 20004
1.49	0.12254 00011	0.96489 78648	1.99	0.02150 82701	0.99511 14132
1.50	0.11893 02892	0.96610 51465	2.00	0.02066 69854	0.99532 22650
	[(−5)1 5]	[(−5)1 5]		[(−5)1 5]	[(−6)4 5]

$$\frac{\sqrt{\pi}}{2} = 0.88622\ 69255$$

Kako napisati program za računanje vrst za Erf(x):

(*)

$$Erf(x,N) \doteq \frac{2}{\sqrt{\pi}} \sum_{n=0}^{N} \underbrace{(-1)^n \frac{z^{2n+1}}{n!(2n+1)}}_{a_n}$$

$$a_0 = 1 \frac{z}{1 \cdot 1} = z$$

$$a_1 = (-1) \cdot \frac{z^3}{1 \cdot 3} = a_0 \cdot (-1) \frac{z^2 \cdot 1}{3 \cdot 1}$$

$$a_2 = (1) \frac{z^5}{2 \cdot 5} : a_1 (-1) \cdot \frac{z^2 \cdot 3}{2 \cdot 5}$$

$$a_3 = (-1) \cdot \frac{z^7}{3 \cdot 2 \cdot 7} = a_2 (-1) \cdot \frac{z^2 \cdot 5}{3 \cdot 7}$$

$$\vdots$$

$$\boxed{a_n = a_{n-1} \cdot (-1) \frac{z^2}{n} \left[\frac{2n-1}{2n+1} \right]}$$

Znova si pomagam z rekurzivno formulo, da se vsakemu računanju potenc!

(**)

$$x \cdot \sqrt{\pi}\, e^{x^2} Erfc(x,N) = 1 + \sum_{n=1}^{N} \underbrace{\frac{(2n-1)!!}{(-2z^2)^n}}_{a_n}$$

$$a_n = \frac{(2n-1)!!}{(-2z^2)^n} \quad ; \quad a_{n-1} = \frac{(2n-3)!!}{(-2z^2)^{n-1}}$$

$$\frac{a_n}{a_{n-1}} = \frac{(2n-1)!! \; (-2z^2)^{n-1}}{(-2z^2)^n \; (2n-3)!!} = \frac{2n-1}{(-2z^2)}$$

$$a_1 = \frac{1}{-2z^2}$$

$$\boxed{a_n = a_{n-1} \cdot \frac{2n-1}{(-2z^2)}}$$

Funkcija $\Gamma(x)$:

- Kako izračunati npr. $\frac{1}{2}!$? Poznamo fakulteto naravnih števil in težavo: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$
Pri računanju fakultete npr. pozitivnih realnih števil pa si pomagamo s funkcijo $\Gamma(x)$, ki je posplošitev funkcije fakulteta na celotna kompleksna števila.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \underbrace{(z>0)}$$

Zahtevamo, da pri negativnih celih številih.

Velja: $n! = \Gamma(n+1)$

- Pri numeričnem računanju določenih integralov si včasih pomagamo s približki, pri čemer uporabimo izraz (Abramowitz Stegun, str. 255)

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \underbrace{\left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right]}_{g(z,n)}$$

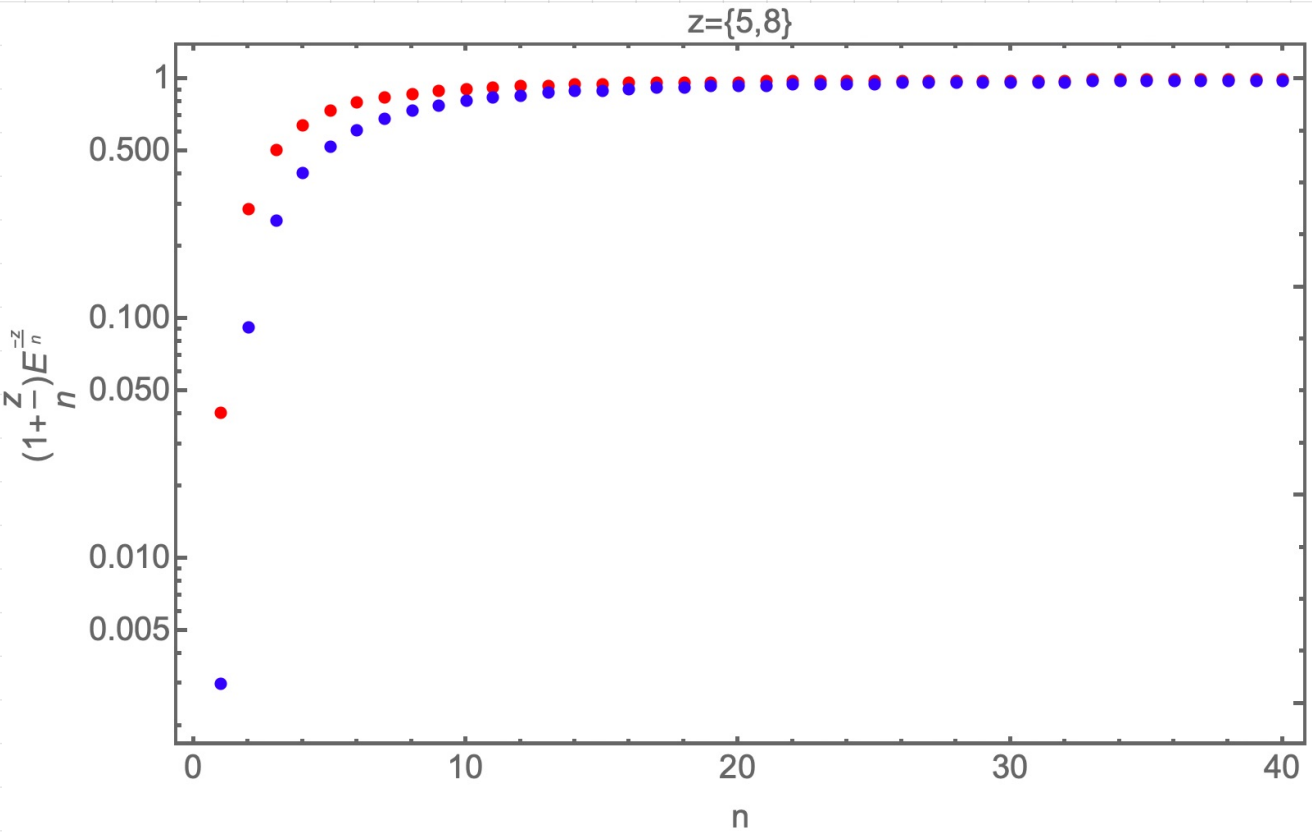
$\gamma = 0.577216$ (Eulerjeva konstanta)

Opažanje, da velja:

$$\lim_{n\to\infty} g(z,n) = \lim_{n\to\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} = 1$$

$\frac{z}{n} \to 0$; $1+\frac{z}{n} \to 1$; $e^{-\frac{z}{n}} \to 1$

Od nekega n naprej bodo člani produkta ≈ 1 in ne bodo več spreminjali končne vrednosti produkta.



Opomba: Vrednost $\frac{1}{2}!$ se da izračunati tudi analitično.

$$\frac{1}{2}! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt = (*)$$

$\Gamma(z+1) = z\,\Gamma(z)$

Uporabimo substitucijo: $t = u^2$, $u = \sqrt{t}$,

$$du = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt \Rightarrow 2\,du = \frac{dt}{\sqrt{t}}$$

$$(*) = \frac{1}{2}\,2\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{2\pi}\,\frac{1}{\sqrt{2}}}{\sqrt{2\pi}\,\frac{1}{\sqrt{2}}}\int_0^\infty e^{-\frac{u^2}{2\left(\frac{1}{\sqrt{2}}\right)^2}}\,du =$$

σ

To prevedemo na integral Gaussove porazdelitve

$= 1$

$$= \frac{1}{2}$$

$$= \sqrt{\pi}\cdot\frac{1}{2} = \frac{\sqrt{\pi}}{2} = \underline{0.886}$$

6. Gamma Function and Related Functions

Mathematical Properties

6.1. Gamma (Factorial) Function

Euler's Integral

6.1.1
$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt \qquad (\Re z>0)$$
$$=k^z\int_0^\infty t^{z-1}e^{-kt}\,dt \quad (\Re z>0,\ \Re k>0)$$

Euler's Formula

6.1.2
$$\Gamma(z)=\lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\ldots(z+n)} \quad (z\neq 0,-1,-2,\ldots)$$

Euler's Infinite Product

6.1.3
$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right] \qquad (|z|<\infty)$$
$$\gamma=\lim_{m\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{m}-\ln m\right]$$
$$=.57721\ 56649\ldots$$

γ is known as Euler's constant and is given to 25 decimal places in chapter 1. $\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z=-n(n=0,\ 1,\ 2,\ \ldots)$ where it possesses simple poles with residue $(-1)^n/n!$. Its reciprocal $1/\Gamma(z)$ is an entire function possessing simple zeros at the points $z=-n(n=0,\ 1,\ 2,\ \ldots)$.

Hankel's Contour Integral

6.1.4
$$\frac{1}{\Gamma(z)}=\frac{i}{2\pi}\int_C(-t)^{-z}e^{-t}\,dt \qquad (|z|<\infty)$$

The path of integration C starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point.

Factorial and Π Notations

6.1.5
$$\Pi(z)=z!=\Gamma(z+1)$$

Integer Values

6.1.6
$$\Gamma(n+1)=1\cdot 2\cdot 3\ldots(n-1)n=n!$$

6.1.7
$$\lim_{z\to n}\frac{1}{\Gamma(-z)}=0=\frac{1}{(-n-1)!} \qquad (n=0,\ 1,\ 2,\ \ldots)$$

Fractional Values

6.1.8
$$\Gamma(\tfrac{1}{2})=2\int_0^\infty e^{-t^2}dt=\pi^{\frac{1}{2}}=1.77245\ 38509\ldots=(-\tfrac{1}{2})!$$

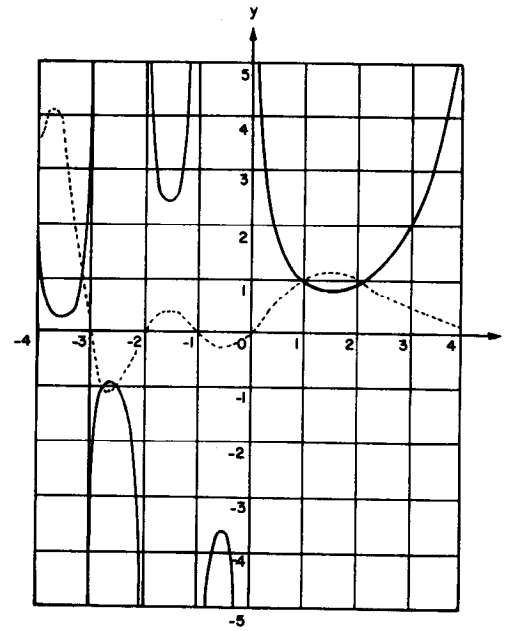


FIGURE 6.1. *Gamma function.* *

———, $y=\Gamma(x)$, – – – –, $y=1/\Gamma(x)$

6.1.9
$$\Gamma(3/2)=\tfrac{1}{2}\pi^{\frac{1}{2}}=.88622\ 69254\ldots=(\tfrac{1}{2})!$$

6.1.10
$$\Gamma(n+\tfrac{1}{4})=\frac{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)}{4^n}\Gamma(\tfrac{1}{4})$$
$$\Gamma(\tfrac{1}{4})=3.62560\ 99082\ldots$$

6.1.11
$$\Gamma(n+\tfrac{1}{3})=\frac{1\cdot 4\cdot 7\cdot 10\ldots(3n-2)}{3^n}\Gamma(\tfrac{1}{3})$$
$$\Gamma(\tfrac{1}{3})=2.67893\ 85347\ldots$$

6.1.12
$$\Gamma(n+\tfrac{1}{2})=\frac{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)}{2^n}\Gamma(\tfrac{1}{2})$$

6.1.13
$$\Gamma(n+\tfrac{2}{3})=\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{3^n}\Gamma(\tfrac{2}{3})$$
$$\Gamma(\tfrac{2}{3})=1.35411\ 79394\ldots$$

6.1.14
$$\Gamma(n+\tfrac{3}{4})=\frac{3\cdot 7\cdot 11\cdot 15\ldots(4n-1)}{4^n}\Gamma(\tfrac{3}{4})$$
$$\Gamma(\tfrac{3}{4})=1.22541\ 67024\ldots$$

*See page II.

Recurrence Formulas

6.1.15 $\Gamma(z+1)=z\Gamma(z)=z!=z(z-1)!$

6.1.16

$$\Gamma(n+z)=(n-1+z)(n-2+z)\ldots(1+z)\Gamma(1+z)$$
$$=(n-1+z)!$$
$$=(n-1+z)(n-2+z)\ldots(1+z)z!$$

Reflection Formula

6.1.17 $\Gamma(z)\Gamma(1-z)=-z\Gamma(-z)\Gamma(z)=\pi\csc\pi z$

$$=\int_0^\infty \frac{t^{z-1}}{1+t}dt \qquad (0<\Re z<1)$$

Duplication Formula

6.1.18 $\Gamma(2z)=(2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\,\Gamma(z+\frac{1}{2})$

Triplication Formula

6.1.19 $\Gamma(3z)=(2\pi)^{-1}3^{3z-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{3})\Gamma(z+\frac{2}{3})$

Gauss' Multiplication Formula

6.1.20 $\Gamma(nz)=(2\pi)^{\frac{1}{2}(1-n)}n^{nz-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)$

Binomial Coefficient

6.1.21 $\binom{z}{w}=\frac{z!}{w!(z-w)!}=\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$

Pochhammer's Symbol

6.1.22

$(z)_0=1,$

$(z)_n=z(z+1)(z+2)\ldots(z+n-1)=\frac{\Gamma(z+n)}{\Gamma(z)}$

Gamma Function in the Complex Plane

6.1.23 $\Gamma(\overline{z})=\overline{\Gamma(z)};\ \ln\Gamma(\overline{z})=\overline{\ln\Gamma(z)}$

6.1.24 $\arg\Gamma(z+1)=\arg\Gamma(z)+\arctan\frac{y}{x}$

6.1.25 $\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right|^2=\prod_{n=0}^{\infty}\left[1+\frac{y^2}{(x+n)^2}\right]^{-1}$

6.1.26 $|\Gamma(x+iy)|\leq|\Gamma(x)|$

6.1.27

$$\arg\Gamma(x+iy)=y\psi(x)+\sum_{n=0}^{\infty}\left(\frac{y}{x+n}-\arctan\frac{y}{x+n}\right)$$
$$(x+iy\neq 0,-1,-2,\ldots)$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$

6.1.28 $\Gamma(1+iy)=iy\,\Gamma(iy)$

6.1.29 $\Gamma(iy)\Gamma(-iy)=|\Gamma(iy)|^2=\frac{\pi}{y\sinh\pi y}$

6.1.30 $\Gamma(\frac{1}{2}+iy)\Gamma(\frac{1}{2}-iy)=|\Gamma(\frac{1}{2}+iy)|^2=\frac{\pi}{\cosh\pi y}$

6.1.31 $\Gamma(1+iy)\Gamma(1-iy)=|\Gamma(1+iy)|^2=\frac{\pi y}{\sinh\pi y}$

6.1.32 $\Gamma(\frac{1}{4}+iy)\Gamma(\frac{3}{4}-iy)=\frac{\pi\sqrt{2}}{\cosh\pi y+i\sinh\pi y}$

Power Series

6.1.33

$$\ln\Gamma(1+z)=-\ln(1+z)+z(1-\gamma)$$
$$+\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]z^n/n \qquad (|z|<2)$$

$\zeta(n)$ is the Riemann Zeta Function (see chapter 23).

Series Expansion[2] for $1/\Gamma(z)$

6.1.34 $\frac{1}{\Gamma(z)}=\sum_{k=1}^{\infty}c_kz^k \qquad (|z|<\infty)$

k	c_k
1	1.00000 00000 000000
2	0.57721 56649 015329
3	−0.65587 80715 202538
4	−0.04200 26350 340952
5	0.16653 86113 822915
6	−0.04219 77345 555443
7	−0.00962 19715 278770
8	0.00721 89432 466630
9	−0.00116 51675 918591
10	−0.00021 52416 741149
11	0.00012 80502 823882
12	−0.00002 01348 547807
13	−0.00000 12504 934821
14	0.00000 11330 272320
15	−0.00000 02056 338417
16	0.00000 00061 160950
17	0.00000 00050 020075
18	−0.00000 00011 812746
19	0.00000 00001 043427
20	0.00000 00000 077823
21	−0.00000 00000 036968
22	0.00000 00000 005100
23	−0.00000 00000 000206
24	−0.00000 00000 000054
25	0.00000 00000 000014
26	0.00000 00000 000001

[2] The coefficients c_k are from H. T. Davis, Tables of higher mathematical functions, 2 vols., Principia Press, Bloomington, Ind., 1933, 1935 (with permission); with corrections due to H. E. Salzer.

GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS Table 6.1

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.000	1.00000 00000	0.00000 00000	-0.57721 56649	1.64493 40668	0.000
1.005	0.99713 85354	-0.00286 55666	-0.56902 09113	1.63299 41567	0.005
1.010	0.99432 58512	-0.00569 03079	-0.56088 54579	1.62121 35283	0.010
1.015	0.99156 12888	-0.00847 45187	-0.55280 85156	1.60958 91824	0.015
1.020	0.98884 42033	-0.01121 84893	-0.54478 93105	1.59811 81919	0.020
1.025	0.98617 39633	-0.01392 25067	-0.53682 70828	1.58679 76993	0.025
1.030	0.98354 99506	-0.01658 68539	-0.52892 10873	1.57562 49154	0.030
1.035	0.98097 15606	-0.01921 18101	-0.52107 05921	1.56459 71163	0.035
1.040	0.97843 82009	-0.02179 76511	-0.51327 48789	1.55371 16426	0.040
1.045	0.97594 92919	-0.02434 46490	-0.50553 32428	1.54296 58968	0.045
1.050	0.97350 42656	-0.02685 30725	-0.49784 49913	1.53235 73421	0.050
1.055	0.97110 25663	-0.02932 31868	-0.49020 94448	1.52188 35001	0.055
1.060	0.96874 36495	-0.03175 52537	-0.48262 59358	1.51154 19500	0.060
1.065	0.96642 69823	-0.03414 95318	-0.47509 38088	1.50133 03259	0.065
1.070	0.96415 20425	-0.03650 62763	-0.46761 24199	1.49124 63164	0.070
1.075	0.96191 83189	-0.03882 57395	-0.46018 11367	1.48128 76622	0.075
1.080	0.95972 53107	-0.04110 81702	-0.45279 93380	1.47145 21556	0.080
1.085	0.95757 25273	-0.04335 38143	-0.44546 64135	1.46173 76377	0.085
1.090	0.95545 94882	-0.04556 29148	-0.43818 17635	1.45214 19988	0.090
1.095	0.95338 57227	-0.04773 57114	-0.43094 47988	1.44266 31755	0.095
1.100	0.95135 07699	-0.04987 24413	-0.42375 49404	1.43329 91508	0.100
1.105	0.94935 41778	-0.05197 33384	-0.41661 16193	1.42404 79514	0.105
1.110	0.94739 55040	-0.05403 86341	-0.40951 42761	1.41490 76482	0.110
1.115	0.94547 43149	-0.05606 85568	-0.40246 23611	1.40587 63535	0.115
1.120	0.94359 01856	-0.05806 33325	-0.39545 53339	1.39695 22213	0.120
1.125	0.94174 26997	-0.06002 31841	-0.38849 26633	1.38813 34449	0.125
1.130	0.93993 14497	-0.06194 83322	-0.38157 38268	1.37941 82573	0.130
1.135	0.93815 60356	-0.06383 89946	-0.37469 83110	1.37080 49288	0.135
1.140	0.93641 60657	-0.06569 53867	-0.36786 56106	1.36229 17670	0.140
1.145	0.93471 11562	-0.06751 77212	-0.36107 52291	1.35387 71152	0.145
1.150	0.93304 09311	-0.06930 62087	-0.35432 66780	1.34555 93520	0.150
1.155	0.93140 50217	-0.07106 10569	-0.34761 94768	1.33733 68900	0.155
1.160	0.92980 30666	-0.07278 24716	-0.34095 31528	1.32920 81752	0.160
1.165	0.92823 47120	-0.07447 06558	-0.33432 72413	1.32117 16859	0.165
1.170	0.92669 96106	-0.07612 58106	-0.32774 12847	1.31322 59322	0.170
1.175	0.92519 74225	-0.07774 81345	-0.32119 48332	1.30536 94548	0.175
1.180	0.92372 78143	-0.07933 78240	-0.31468 74438	1.29760 08248	0.180
1.185	0.92229 04591	-0.08089 50733	-0.30821 86809	1.28991 86421	0.185
1.190	0.92088 50371	-0.08242 00745	-0.30178 81156	1.28232 15358	0.190
1.195	0.91951 12341	-0.08391 30174	-0.29539 53259	1.27480 81622	0.195
1.200	0.91816 87424	-0.08537 40900	-0.28903 98966	1.26737 72054	0.200
1.205	0.91685 72606	-0.08680 34780	-0.28272 14187	1.26002 73755	0.205
1.210	0.91557 64930	-0.08820 13651	-0.27643 94897	1.25275 74090	0.210
1.215	0.91432 61500	-0.08956 79331	-0.27019 37135	1.24556 60671	0.215
1.220	0.91310 59475	-0.09090 33619	-0.26398 37000	1.23845 21360	0.220
1.225	0.91191 56071	-0.09220 78291	-0.25780 90652	1.23141 44258	0.225
1.230	0.91075 48564	-0.09348 15108	-0.25166 94307	1.22445 17702	0.230
1.235	0.90962 34274	-0.09472 45811	-0.24556 44243	1.21756 30254	0.235
1.240	0.90852 10583	-0.09593 72122	-0.23949 36791	1.21074 70707	0.240
1.245	0.90744 74922	-0.09711 95744	-0.23345 68341	1.20400 28063	0.245
1.250	0.90640 24771	-0.09827 18364	-0.22745 35334	1.19732 91545	0.250
	$y!$	$\ln y!$	$\frac{d}{dy} \ln y!$	$\frac{d^2}{dy^2} \ln y!$	y
	$\left[\begin{matrix}(-6)6\\5\end{matrix}\right]$	$\left[\begin{matrix}(-6)5\\5\end{matrix}\right]$	$\left[\begin{matrix}(-6)7\\5\end{matrix}\right]$	$\left[\begin{matrix}(-5)2\\5\end{matrix}\right]$	

$\log_{10} e = 0.43429\ 44819$

For $x>2$ see **Examples 1–4.**

Compiled from H. T. Davis, Tables of the higher mathematical functions, 2 vols. (Principia Press, Bloomington, Ind., 1933, 1935) (with permission). Known error has been corrected.

Table 6.1 GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.250	0.90640 24771	-0.09827 18364	-0.22745 35334	1.19732 91545	0.250
1.255	0.90538 57663	-0.09939 41651	-0.22148 34266	1.19072 50579	0.255
1.260	0.90439 71178	-0.10048 67254	-0.21554 61686	1.18418 94799	0.260
1.265	0.90343 62946	-0.10154 96809	-0.20964 14193	1.17772 14030	0.265
1.270	0.90250 30645	-0.10258 31932	-0.20376 88437	1.17131 98301	0.270
1.275	0.90159 71994	-0.10358 74224	-0.19792 81118	1.16498 37821	0.275
1.280	0.90071 84765	-0.10456 25269	-0.19211 88983	1.15871 22990	0.280
1.285	0.89986 66769	-0.10550 86634	-0.18634 08828	1.15250 44385	0.285
1.290	0.89904 15863	-0.10642 59872	-0.18059 37494	1.14635 92764	0.290
1.295	0.89824 29947	-0.10731 46519	-0.17487 71870	1.14027 59053	0.295
1.300	0.89747 06963	-0.10817 48095	-0.16919 08889	1.13425 34350	0.300
1.305	0.89672 44895	-0.10900 66107	-0.16353 45526	1.12829 09915	0.305
1.310	0.89600 41767	-0.10981 02045	-0.15790 78803	1.12238 77175	0.310
1.315	0.89530 95644	-0.11058 57384	-0.15231 05782	1.11654 27706	0.315
1.320	0.89464 04630	-0.11133 33587	-0.14674 23568	1.11075 53246	0.320
1.325	0.89399 66866	-0.11205 32100	-0.14120 29305	1.10502 45678	0.325
1.330	0.89337 80535	-0.11274 54356	-0.13569 20180	1.09934 97037	0.330
1.335	0.89278 43850	-0.11341 01772	-0.13020 93416	1.09372 99497	0.335
1.340	0.89221 55072	-0.11404 75756	-0.12475 46279	1.08816 45379	0.340
1.345	0.89167 12485	-0.11465 77697	-0.11932 76069	1.08265 27136	0.345
1.350	0.89115 14420	-0.11524 08974	-0.11392 80127	1.07719 37361	0.350
1.355	0.89065 59235	-0.11579 70951	-0.10855 55827	1.07178 68773	0.355
1.360	0.89018 45324	-0.11632 64980	-0.10321 00582	1.06643 14226	0.360
1.365	0.88973 71116	-0.11682 92401	-0.09789 11840	1.06112 66696	0.365
1.370	0.88931 35074	-0.11730 54539	-0.09259 87082	1.05587 19286	0.370
1.375	0.88891 35692	-0.11775 52707	-0.08733 23825	1.05066 65216	0.375
1.380	0.88853 71494	-0.11817 88209	-0.08209 19619	1.04550 97829	0.380
1.385	0.88818 41041	-0.11857 62331	-0.07687 72046	1.04040 10578	0.385
1.390	0.88785 42918	-0.11894 76353	-0.07168 78723	1.03533 97036	0.390
1.395	0.88754 75748	-0.11929 31538	-0.06652 37297	1.03032 50881	0.395
1.400	0.88726 38175	-0.11961 29142	-0.06138 45446	1.02535 65905	0.400
1.405	0.88700 28884	-0.11990 70405	-0.05627 00879	1.02043 36002	0.405
1.410	0.88676 46576	-0.12017 56559	-0.05118 01337	1.01555 55173	0.410
1.415	0.88654 89993	-0.12041 88823	-0.04611 44589	1.01072 17518	0.415
1.420	0.88635 57896	-0.12063 68406	-0.04107 28433	1.00593 17241	0.420
1.425	0.88618 49081	-0.12082 96505	-0.03605 50697	1.00118 48640	0.425
1.430	0.88603 62361	-0.12099 74307	-0.03106 09237	0.99648 06113	0.430
1.435	0.88590 96587	-0.12114 02987	-0.02609 01935	0.99181 84147	0.435
1.440	0.88580 50635	-0.12125 83713	-0.02114 26703	0.98719 77326	0.440
1.445	0.88572 23397	-0.12135 17638	-0.01621 81479	0.98261 80318	0.445
1.450	0.88566 13803	-0.12142 05907	-0.01131 64226	0.97807 87886	0.450
1.455	0.88562 20800	-0.12146 49657	-0.00643 72934	0.97357 94874	0.455
1.460	0.88560 43364	-0.12148 50010	-0.00158 05620	0.96911 96215	0.460
1.465	0.88560 80495	-0.12148 08083	+0.00325 39677	0.96469 86921	0.465
1.470	0.88563 31217	-0.12145 24980	0.00806 64890	0.96031 62091	0.470
1.475	0.88567 94575	-0.12140 01797	0.01285 71930	0.95597 16896	0.475
1.480	0.88574 69646	-0.12132 39621	0.01762 62684	0.95166 46592	0.480
1.485	0.88583 55520	-0.12122 39528	0.02237 39013	0.94739 46509	0.485
1.490	0.88594 51316	-0.12110 02585	0.02710 02758	0.94316 12052	0.490
1.495	0.88607 56174	-0.12095 29852	0.03180 55736	0.93896 38700	0.495
1.500	0.88622 69255	-0.12078 22376	0.03648 99740	0.93480 22005	0.500
	$y!$	$\ln y!$	* $\frac{d}{dy} \ln y!$	* $\frac{d^2}{dy^2} \ln y!$	y
	$\left[\begin{smallmatrix}(-6)4\\5\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)4\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)4\\5\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)9\\5\end{smallmatrix}\right]$	

$\log_{10} e = 0.43429\ 44819$

*See page 11.

GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS Table 6.1

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.500	0.88622 69255	-0.12078 22376	0.03648 99740	0.93480 22005	0.500
1.505	0.88639 89744	-0.12058 81200	0.04115 36543	0.93067 57588	0.505
1.510	0.88659 16850	-0.12037 07353	0.04579 67896	0.92658 41142	0.510
1.515	0.88680 49797	-0.12013 01860	0.05041 95527	0.92252 68425	0.515
1.520	0.88703 87833	-0.11986 65735	0.05502 21146	0.91850 35265	0.520
1.525	0.88729 30231	-0.11957 99983	0.05960 46439	0.91451 37552	0.525
1.530	0.88756 76278	-0.11927 05601	0.06416 73074	0.91055 71245	0.530
1.535	0.88786 25287	-0.11893 83580	0.06871 02697	0.90663 32361	0.535
1.540	0.88817 76586	-0.11858 34900	0.07323 36936	0.90274 16984	0.540
1.545	0.88851 29527	-0.11820 60534	0.07773 77400	0.89888 21253	0.545
1.550	0.88886 83478	-0.11780 61446	0.08222 25675	0.89505 41371	0.550
1.555	0.88924 37830	-0.11738 38595	0.08668 83334	0.89125 73596	0.555
1.560	0.88963 91990	-0.11693 92928	0.09113 51925	0.88749 14249	0.560
1.565	0.89005 45387	-0.11647 25388	0.09556 32984	0.88375 59699	0.565
1.570	0.89048 97463	-0.11598 36908	0.09997 28024	0.88005 06378	0.570
1.575	0.89094 47686	-0.11547 28415	0.10436 38544	0.87637 50766	0.575
1.580	0.89141 95537	-0.11494 00828	0.10873 66023	0.87272 89402	0.580
1.585	0.89191 40515	-0.11438 55058	0.11309 11923	0.86911 18871	0.585
1.590	0.89242 82141	-0.11380 92009	0.11742 77690	0.86552 35815	0.590
1.595	0.89296 19949	-0.11321 12579	0.12174 64754	0.86196 36921	0.595
1.600	0.89351 53493	-0.11259 17657	0.12604 74528	0.85843 18931	0.600
1.605	0.89408 82342	-0.11195 08127	0.13033 08407	0.85492 78630	0.605
1.610	0.89468 06085	-0.11128 84864	0.13459 67772	0.85145 12856	0.610
1.615	0.89529 24327	-0.11060 48737	0.13884 53988	0.84800 18488	0.615
1.620	0.89592 36685	-0.10990 00610	0.14307 68404	0.84457 92455	0.620
1.625	0.89657 42800	-0.10917 41338	0.14729 12354	0.84118 31730	0.625
1.630	0.89724 42326	-0.10842 71769	0.15148 87158	0.83781 33330	0.630
1.635	0.89793 34930	-0.10765 92746	0.15566 94120	0.83446 94315	0.635
1.640	0.89864 20302	-0.10687 05105	0.15983 34529	0.83115 11790	0.640
1.645	0.89936 98138	-0.10606 09676	0.16398 09660	0.82785 82897	0.645
1.650	0.90011 68163	-0.10523 07282	0.16811 20776	0.82459 04826	0.650
1.655	0.90088 30104	-0.10437 98739	0.17222 69122	0.82134 74802	0.655
1.660	0.90166 83712	-0.10350 84860	0.17632 55933	0.81812 90092	0.660
1.665	0.90247 28748	-0.10261 66447	0.18040 82427	0.81493 48001	0.665
1.670	0.90329 64995	-0.10170 44301	0.18447 49813	0.81176 45875	0.670
1.675	0.90413 92243	-0.10077 19212	0.18852 59282	0.80861 81094	0.675
1.680	0.90500 10302	-0.09981 91969	0.19256 12015	0.80549 51079	0.680
1.685	0.90588 18996	-0.09884 63351	0.19658 09180	0.80239 53282	0.685
1.690	0.90678 18160	-0.09785 34135	0.20058 51931	0.79931 85198	0.690
1.695	0.90770 07650	-0.09684 05088	0.20457 41410	0.79626 44350	0.695
1.700	0.90863 87329	-0.09580 76974	0.20854 78749	0.79323 28302	0.700
1.705	0.90959 57079	-0.09475 50552	0.21250 65064	0.79022 34645	0.705
1.710	0.91057 16796	-0.09368 26573	0.21645 01462	0.78723 61012	0.710
1.715	0.91156 66390	-0.09259 05785	0.22037 89037	0.78427 05060	0.715
1.720	0.91258 05779	-0.09147 88929	0.22429 28871	0.78132 64486	0.720
1.725	0.91361 34904	-0.09034 76741	0.22819 22037	0.77840 37011	0.725
1.730	0.91466 53712	-0.08919 69951	0.23207 69593	0.77550 20396	0.730
1.735	0.91573 62171	-0.08802 69286	0.23594 72589	0.77262 12424	0.735
1.740	0.91682 60252	-0.08683 75466	0.23980 32061	0.76976 10915	0.740
1.745	0.91793 47950	-0.08562 89203	0.24364 49038	0.76692 13714	0.745
1.750	0.91906 25268	-0.08440 11210	0.24747 24535	0.76410 18699	0.750
	$y!$	$\ln y!$	$\frac{d}{dy} \ln y!$	$\frac{d^2}{dy^2} \ln y!$	y
	$\left[\begin{smallmatrix}(-6)3\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)3\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)3\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)4\\5\end{smallmatrix}\right]$	

$\log_{10} e = 0.43429\ 44819$

Table 6.1 GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.750	0.91906 25268	-0.08440 11210	0.24747 24535	0.76410 18699	0.750
1.755	0.92020 92224	-0.08315 42192	0.25128 59559	0.76130 23773	0.755
1.760	0.92137 48846	-0.08188 82847	0.25508 55103	0.75852 26870	0.760
1.765	0.92255 95178	-0.08060 33871	0.25887 12154	0.75576 25950	0.765
1.770	0.92376 31277	-0.07929 95955	0.26264 31686	0.75302 19003	0.770
1.775	0.92498 57211	-0.07797 69782	0.26640 14664	0.75030 04040	0.775
1.780	0.92622 73062	-0.07663 56034	0.27014 62043	0.74759 79107	0.780
1.785	0.92748 78926	-0.07527 55386	0.27387 74769	0.74491 42268	0.785
1.790	0.92876 74904	-0.07389 68509	0.27759 53776	0.74224 91617	0.790
1.795	0.93006 61123	-0.07249 96070	0.28129 99992	0.73960 25271	0.795
1.800	0.93138 37710	-0.07108 38729	0.28499 14333	0.73697 41375	0.800
1.805	0.93272 04811	-0.06964 97145	0.28866 97707	0.73436 38093	0.805
1.810	0.93407 62585	-0.06819 71969	0.29233 51012	0.73177 13620	0.810
1.815	0.93545 11198	-0.06672 63850	0.29598 75138	0.72919 66166	0.815
1.820	0.93684 50832	-0.06523 73431	0.29962 70966	0.72663 93972	0.820
1.825	0.93825 81682	-0.06373 01353	0.30325 39367	0.72409 95297	0.825
1.830	0.93969 03951	-0.06220 48248	0.30686 81205	0.72157 68426	0.830
1.835	0.94114 17859	-0.06066 14750	0.31046 97335	0.71907 11662	0.835
1.840	0.94261 23634	-0.05910 01483	0.31405 88602	0.71658 23333	0.840
1.845	0.94410 21519	-0.05752 09071	0.31763 55846	0.71411 01788	0.845
1.850	0.94561 11764	-0.05592 38130	0.32119 99895	0.71165 45396	0.850
1.855	0.94713 94637	-0.05430 89276	0.32475 21572	0.70921 52546	0.855
1.860	0.94868 70417	-0.05267 63117	0.32829 21691	0.70679 21650	0.860
1.865	0.95025 39389	-0.05102 60260	0.33182 01056	0.70438 51138	0.865
1.870	0.95184 01855	-0.04935 81307	0.33533 60467	0.70199 39461	0.870
1.875	0.95344 58127	-0.04767 26854	0.33884 00713	0.69961 85089	0.875
1.880	0.95507 08530	-0.04596 97497	0.34233 22577	0.69725 86512	0.880
1.885	0.95671 53398	-0.04424 93824	0.34581 26835	0.69491 42236	0.885
1.890	0.95837 93077	-0.04251 16423	0.34928 14255	0.69258 50790	0.890
1.895	0.96006 27927	-0.04075 65875	0.35273 85596	0.69027 10717	0.895
1.900	0.96176 58319	-0.03898 42759	0.35618 41612	0.68797 20582	0.900
1.905	0.96348 84632	-0.03719 47650	0.35961 83049	0.68568 78965	0.905
1.910	0.96523 07261	-0.03538 81118	0.36304 10646	0.68341 84465	0.910
1.915	0.96699 26608	-0.03356 43732	0.36645 25136	0.68116 35696	0.915
1.920	0.96877 43090	-0.03172 36054	0.36985 27244	0.67892 31293	0.920
1.925	0.97057 57134	-0.02986 58646	0.37324 17688	0.67669 69903	0.925
1.930	0.97239 69178	-0.02799 12062	0.37661 97179	0.67448 50194	0.930
1.935	0.97423 79672	-0.02609 96858	0.37998 66424	0.67228 70846	0.935
1.940	0.97609 89075	-0.02419 13581	0.38334 26119	0.67010 30559	0.940
1.945	0.97797 97861	-0.02226 62778	0.38668 76959	0.66793 28044	0.945
1.950	0.97988 06513	-0.02032 44991	0.39002 19627	0.66577 62034	0.950
1.955	0.98180 15524	-0.01836 60761	0.39334 54805	0.66363 31270	0.955
1.960	0.98374 25404	-0.01639 10621	0.39665 83163	0.66150 34514	0.960
1.965	0.98570 36664	-0.01439 95106	0.39996 05371	0.65938 70538	0.965
1.970	0.98768 49838	-0.01239 14744	0.40325 22088	0.65728 38134	0.970
1.975	0.98968 65462	-0.01036 70060	0.40653 33970	0.65519 36104	0.975
1.980	0.99170 84087	-0.00832 61578	0.40980 41664	0.65311 63266	0.980
1.985	0.99375 06274	-0.00626 89816	0.41306 45816	0.65105 18450	0.985
1.990	0.99581 32598	-0.00419 55291	0.41631 47060	0.64900 00505	0.990
1.995	0.99789 63643	-0.00210 58516	0.41955 46030	0.64696 08286	0.995
2.000	1.00000 00000	0.00000 00000	0.42278 43351	0.64493 40668	1.000
	$y!$	$\ln y!$	$\frac{d}{dy} \ln y!$	$\frac{d^2}{dy^2} \ln y!$	y
	$\left[\begin{smallmatrix}(-6)2\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)2\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)2\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)2\\4\end{smallmatrix}\right]$	

$\log_{10} e = 0.43429\ 44819$

TETRAGAMMA AND PENTAGAMMA FUNCTIONS

Table 6.2

x	$\psi''(x)$	$\psi^{(3)}(x)$		x	$\psi''(x)$	$\psi^{(3)}(x)$	
1.00	-2.40411 38063	6.49393 94023	0.00	1.50	-0.82879 66442	1.40909 10340	0.50
1.01	-2.34039 86771	6.25106 18729	0.01	1.51	-0.81487 76121	1.37489 70527	0.51
1.02	-2.27905 42052	6.01969 49890	0.02	1.52	-0.80129 51399	1.34177 21104	0.52
1.03	-2.21996 85963	5.79918 38573	0.03	1.53	-0.78803 87419	1.30967 56244	0.53
1.04	-2.16303 63855	5.58891 68399	0.04	1.54	-0.77509 83287	1.27856 88154	0.54
1.05	-2.10815 80219	5.38832 23132	0.05	1.55	-0.76246 41904	1.24841 46160	0.55
1.06	-2.05523 94833	5.19686 56970	0.06	1.56	-0.75012 69793	1.21917 75841	0.56
1.07	-2.00419 19194	5.01404 67303	0.07	1.57	-0.73807 76946	1.19082 38216	0.57
1.08	-1.95493 13213	4.83939 69702	0.08	1.58	-0.72630 76669	1.16332 08979	0.58
1.09	-1.90737 82154	4.67247 74947	0.09	1.59	-0.71480 85441	1.13663 77770	0.59
1.10	-1.86145 73783	4.51287 67903	0.10	1.60	-0.70357 22779	1.11074 47490	0.60
1.11	-1.81709 75731	4.36020 88083	0.11	1.61	-0.69259 11105	1.08561 33658	0.61
1.12	-1.77423 13035	4.21411 11755	0.12	1.62	-0.68185 75627	1.06121 63792	0.62
1.13	-1.73279 45852	4.07424 35447	0.13	1.63	-0.67136 44220	1.03752 76835	0.63
1.14	-1.69272 67342	3.94028 60737	0.14	1.64	-0.66110 47316	1.01452 22608	0.64
1.15	-1.65397 01677	3.81193 80220	0.15	1.65	-0.65107 17793	0.99217 61290	0.65
1.16	-1.61647 02206	3.68891 64540	0.16	1.66	-0.64125 90881	0.97046 62927	0.66
1.17	-1.58017 49731	3.57095 50416	0.17	1.67	-0.63166 04061	0.94937 06973	0.67
1.18	-1.54503 50903	3.45780 29554	0.18	1.68	-0.62226 96973	0.92886 81843	0.68
1.19	-1.51100 36723	3.34922 38402	0.19	1.69	-0.61308 11332	0.90893 84502	0.69
1.20	-1.47803 61144	3.24499 48647	0.20	1.70	-0.60408 90841	0.88956 20066	0.70
1.21	-1.44608 99765	3.14490 58422	0.21	1.71	-0.59528 81112	0.87072 01433	0.71
1.22	-1.41512 48602	3.04875 84139	0.22	1.72	-0.58667 29593	0.85239 48922	0.72
1.23	-1.38510 22950	2.95636 52925	0.23	1.73	-0.57823 85490	0.83456 89940	0.73
1.24	-1.35598 56308	2.86754 95589	0.24	1.74	-0.56997 99702	0.81722 58660	0.74
1.25	-1.32773 99375	2.78214 40092	0.25	1.75	-0.56189 24756	0.80034 95719	0.75
1.26	-1.30033 19112	2.69999 05478	0.26	1.76	-0.55397 14738	0.78392 47929	0.76
1.27	-1.27372 97857	2.62093 96227	0.27	1.77	-0.54621 25238	0.76793 68005	0.77
1.28	-1.24790 32496	2.54484 97000	0.28	1.78	-0.53861 13291	0.75237 14300	0.78
1.29	-1.22282 33691	2.47158 67746	0.29	1.79	-0.53116 37320	0.73721 50564	0.79
1.30	-1.19846 25147	2.40102 39143	0.30	1.80	-0.52386 57084	0.72245 45705	0.80
1.31	-1.17479 42923	2.33304 08348	0.31	1.81	-0.51671 33630	0.70807 73565	0.81
1.32	-1.15179 34794	2.26752 35032	0.32	1.82	-0.50970 29242	0.69407 12710	0.82
1.33	-1.12943 59642	2.20436 37678	0.33	1.83	-0.50283 07396	0.68042 46226	0.83
1.34	-1.10769 86881	2.14345 90132	0.34	1.84	-0.49609 32712	0.66712 61527	0.84
1.35	-1.08655 95925	2.08471 18367	0.35	1.85	-0.48948 70921	0.65416 50169	0.85
1.36	-1.06599 75682	2.02802 97472	0.36	1.86	-0.48300 88813	0.64153 07680	0.86
1.37	-1.04599 24073	1.97332 48830	0.37	1.87	-0.47665 54207	0.62921 33389	0.87
1.38	-1.02652 47586	1.92051 37473	0.38	1.88	-0.47042 35909	0.61720 30270	0.88
1.39	-1.00757 60850	1.86951 69616	0.39	1.89	-0.46431 03677	0.60549 04793	0.89
1.40	-0.98912 86236	1.82025 90339	0.40	1.90	-0.45831 28188	0.59406 66772	0.90
1.41	-0.97116 53479	1.77266 81419	0.41	1.91	-0.45242 81007	0.58292 29238	0.91
1.42	-0.95366 99322	1.72667 59295	0.42	1.92	-0.44665 34549	0.57205 08299	0.92
1.43	-0.93662 67177	1.68221 73161	0.43	1.93	-0.44098 62055	0.56144 23020	0.93
1.44	-0.92002 06808	1.63923 03178	0.44	1.94	-0.43542 37563	0.55108 95304	0.94
1.45	-0.90383 74031	1.59765 58792	0.45	1.95	-0.42996 35876	0.54098 49774	0.95
1.46	-0.88806 30426	1.55743 77157	0.46	1.96	-0.42460 32537	0.53112 13668	0.96
1.47	-0.87268 43070	1.51852 21649	0.47	1.97	-0.41934 03805	0.52149 16733	0.97
1.48	-0.85768 84281	1.48085 80478	0.48	1.98	-0.41417 26631	0.51208 91127	0.98
1.49	-0.84306 31376	1.44439 65370	0.49	1.99	-0.40909 78630	0.50290 71324	0.99
1.50	-0.82879 66442	1.40909 10340	0.50	2.00	-0.40411 38063	0.49393 94023	1.00
*	$\frac{d^3}{dy^3}\ln y!$	$\frac{d^4}{dy^4}\ln y!$	y		$\frac{d^3}{dy^3}\ln y!$	$\frac{d^4}{dy^4}\ln y!$	y
	$\left[\begin{matrix}(-4)3\\7\end{matrix}\right]$	$\left[\begin{matrix}(-3)1\\7\end{matrix}\right]$			$\left[\begin{matrix}(-5)4\\6\end{matrix}\right]$	$\left[\begin{matrix}(-4)1\\6\end{matrix}\right]$	

Compiled from H. T. Davis, Tables of the higher mathematical functions, 2 vols. (Principia Press, Bloomington, Ind., 1933, 1935) (with permission).

*See page II.